Module 1

Junior Secondary Mathematics

Number Systems

THE COMMONWEALTH *of* LEARNING

Science, Technology and Mathematics Modules

for Upper Primary and Junior Secondary School Teachers of Science, Technology and Mathematics by Distance in the Southern African Development Community (SADC)

Developed by
The Southern African Development Community (SADC)

Ministries of Education in:

- **Botswana**
- **Malawi**
- **Mozambique**
- **Namibia**
- **South Africa**
- **Tanzania**
- **Zambia**
- **Zimbabwe**

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SCIENCE, TECHNOLOGY AND MATHEMATICS MODULES

This module is one of a series prepared under the auspices of the participating Southern African Development Community (SADC) and The Commonwealth of Learning as part of the Training of Upper Primary and Junior Secondary Science, Technology and Mathematics Teachers in Africa by Distance. These modules enable teachers to enhance their professional skills through distance and open learning. Many individuals and groups have been involved in writing and producing these modules. We trust that they will benefit not only the teachers who use them, but also, ultimately, their students and the communities and nations in which they live.

The twenty-eight Science, Technology and Mathematics modules are as follows:

Upper Primary Science
Module 1: *My Built Environment*
Module 2: *Materials in my Environment*
Module 3: *My Health*
Module 4: *My Natural Environment*

Junior Secondary Science
Module 1: *Energy and Energy Transfer*
Module 2: *Energy Use in Electronic Communication*
Module 3: *Living Organisms' Environment and Resources*
Module 4: *Scientific Processes*

Upper Primary Technology
Module 1: *Teaching Technology in the Primary School*
Module 2: *Making Things Move*
Module 3: *Structures*
Module 4: *Materials*
Module 5: *Processing*

Junior Secondary Technology
Module 1: *Introduction to Teaching Technology*
Module 2: *Systems and Controls*
Module 3: *Tools and Materials*
Module 4: *Structures*

Upper Primary Mathematics
Module 1: *Number and Numeration*
Module 2: *Fractions*
Module 3: *Measures*
Module 4: *Social Arithmetic*
Module 5: *Geometry*

Junior Secondary Mathematics
Module 1: *Number Systems*
Module 2: *Number Operations*
Module 3: *Shapes and Sizes*
Module 4: *Algebraic Processes*
Module 5: *Solving Equations*
Module 6: *Data Handling*

A MESSAGE FROM THE COMMONWEALTH OF LEARNING

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Dato' Professor Gajaraj Dhanarajan
President and Chief Executive Officer

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CONTACTS FOR THE PROGRAMME

The Commonwealth of Learning
1285 West Broadway, Suite 600
Vancouver, BC V6H 3X8
Canada

Ministry of Education
Private Bag 005
Gaborone
Botswana

Ministry of Education
Private Bag 328
Capital City
Lilongwe 3
Malawi

Ministério da Eduação
Avenida 24 de Julho No 167, 8
Caixa Postal 34
Maputo
Mozambique

Ministry of Basic Education,
Sports and Culture
Private Bag 13186
Windhoek
Namibia

National Ministry of Education
Private Bag X603
Pretoria 0001
South Africa

Ministry of Education and Culture
P.O. Box 9121
Dar es Salaam
Tanzania

Ministry of Education
P.O. Box 50093
Lusaka
Zambia

Ministry of Education, Sport and Culture
P.O. Box CY 121
Causeway
Harare
Zimbabwe

COURSE WRITERS FOR JUNIOR SECONDARY MATHEMATICS

Ms. Sesutho Koketso Kesianye: *Writing Team Leader*
Head of Mathematics Department
Tonota College of Education
Botswana

Mr. Jan Durwaarder: Lecturer (Mathematics)
Tonota College of Education
Botswana

Mr. Kutengwa Thomas Sichinga: Teacher (Mathematics)
Moshupa Secondary School
Botswana

FACILITATORS/RESOURCE PERSONS

Mr. Bosele Radipotsane: Principal Education Officer (Mathematics)
Ministry of Education
Botswana

Ms. Felicity M Leburu-Sianga: Chief Education Officer
Ministry of Education
Botswana

PROJECT MANAGEMENT & DESIGN

Ms. Kgomotso Motlotle: Education Specialist, Teacher Training
The Commonwealth of Learning (COL)
Vancouver, BC, Canada

Mr. David Rogers: *Post-production Editor*
Open Learning Agency
Victoria, BC, Canada

Ms. Sandy Reber: *Graphics & desktop publishing*
Reber Creative
Victoria, BC, Canada

Introduction

Welcome to the programme in Teaching Junior Secondary Mathematics! This series of six modules is designed to help you to strengthen your knowledge of mathematics topics and to acquire more instructional strategies for teaching mathematics in the classroom.

Each of the six modules in the *Junior Secondary Mathematics* series provides an opportunity to apply theory to practice. Learning about mathematics entails the development of practical skills as well as theoretical knowledge. Each mathematics topic includes an explanation of the theory behind the mathematics, examples of how the mathematics is used in practice, and suggestions for classroom activities that allow students to explore the mathematics for themselves.

Each module also explores several instructional strategies that can be used in the mathematics classroom and provides you with an opportunity to apply these strategies in practical classroom activities. Each module examines the reasons for using a particular strategy in the classroom and provides a guide for the best use of each strategy, given the topic, context and goals.

The guiding principles of these modules are to help make the connection between theoretical maths and the use of the maths; to apply instructional theory to practice in the classroom situation; and to support you, as you in turn help your students to apply mathematics theory to practical classroom work.

Programme Goals

This programme is designed to help you to:

- strengthen your understanding of mathematics topics
- expand the range of instructional strategies that you can use in the mathematics classroom

Programme Objectives

By the time you have completed this programme, you should be able to:

- develop and present lessons on the nature of the mathematics process, with an emphasis on where each type of mathematics is used outside of the classroom
- guide students as they work in teams on practical projects in mathematics, and help them to work effectively as a member of a group
- use questioning and explanation strategies to help students learn new concepts and to support students in their problem solving activities
- guide students in the use of investigative strategies on particular projects, and thus to show them how mathematical tools are used
- guide students as they prepare their portfolios about their project activities

The relationship between this programme and your local mathematics curriculum

The mathematics content presented in these modules includes some of the topics most commonly covered in the mathematics curricula in southern African countries. However, it is not intended to cover all topics in any one country's mathematics curriculum comprehensively. For this, you will need to consult your national or regional curriculum guide. The curriculum content that is presented in these modules is intended to:

- provide an overview of the content in order to support the development of appropriate teaching strategies
- use selected parts of the curriculum as examples for application of specific teaching strategies
- explain those elements of the curriculum that provide essential background knowledge, or that address particularly complex or specialised concepts
- provide directions to additional resources on the curriculum content

How to Work on this Programme

As is indicated in the programme goals and objectives, the programme provides for you to participate actively in each module by applying instructional strategies when exploring mathematics with your students and by reflecting on that experience. In other words, you "put on your student uniform" for the time you work on this course. There are several different ways of doing this.

Working on your own

You may be the only teacher of mathematics in your school, or you may choose to work on your own so you can accommodate this programme within your schedule. If this is the case, these are the recommended strategies for using this module:

1. Establish a schedule for working on the module: choose a date by which you plan to complete the first module, taking into account that each *unit* will require between six to eight hours of study time and about two hours of classroom time for implementing your lesson plan. (Also note that each module contains two to four units.) For example, if you have two hours a week available for study, then each unit will take between three and four weeks to complete. If you have four hours a week for study, then each unit will take about two weeks to complete.

2. Choose a study space where you can work quietly without interruption, for example, a space in your school where you can work after hours.

3. If possible, identify someone who is interested in mathematics or whose interests are relevant to mathematics (for example, a science teacher in your school) with whom you can discuss the module and some of your ideas about teaching mathematics. Even the most independent learner benefits from good dialogue with others: it helps us to formulate our ideas—or as one learner commented, "How do I know what I'm thinking until I hear what I have to say?"

Working with colleagues

If you are in a situation where there are other teachers of mathematics in your school or in your immediate area, then it is possible for you to work together on this module. You may choose to do this informally, perhaps having a discussion group once a week or once every two weeks about a particular topic in one of the units. Or, you may choose to organise more formally, establishing a schedule so that everyone is working on the same units at the same time, and you can work in small groups or pairs on particular projects. If you and several colleagues plan to work together on these modules, these are the recommended steps:

1. Establish and agree on a schedule that allows sufficient time to work on each unit, but also maintains the momentum so that people don't lose interest. If all of you work together in the same location, meeting once a week and allocating two weeks for each unit, this plan should accommodate individual and group study time. If you work in different locations, and have to travel some distance to meet, then you may decide to meet once every two weeks, and agree to complete a unit every two weeks.

2. Develop and agree on group goals, so that everyone is clear about the intended achievements for each unit and for each group session.

3. Develop a plan for each session, outlining what topics will be covered and what activities will be undertaken by the group as a whole, in pairs or in small groups. It may be helpful for each member of the group to take a turn in planning a session.

Your group may also choose to call on the expertise of others, perhaps inviting someone with particular knowledge about teaching or about a specific mathematics topic, to speak with the group.

Your group may also have the opportunity to consult with a mentor, or with other groups.

Colleagues as feedback/resource persons

Even if your colleagues are not participating directly in this programme, they may be interested in hearing about it and about some of your ideas as a result of taking part. Your fellow teachers of mathematics may also be willing to take part in discussions with you about the programme.

Working with a mentor

As mentioned above, you may have the opportunity to work with a mentor, someone with expertise in mathematics education who can provide you with feedback about your work. If you are working on your own, your communication with your mentor may be by letter mail, telephone or e-mail. If you are working as a group, you may have occasional group meetings or teleconferences with your mentor.

Using a learning journal

Whether you are working on your own or with a group, use a learning journal. The learning journal serves a number of different purposes, and you can divide your journal into compartments to accommodate these purposes. You can think of your journal as a "place" with several "rooms" where you can think out loud by writing down your ideas and thoughts.

In one part of your journal, keep your assignments and lesson plans. In another, you can:

- keep notes and a running commentary about what you are reading in each unit
- write down ideas that occur to you about something in the unit, and note questions about the content or anything with which you disagree
- record ideas about how to use some of the content and strategies in the classroom
- record your answers to the "Reflection" questions that occur in most units

If you consistently keep these notes as you work through each unit, they will serve as a resource when you work in the classroom, since you will have already put together some ideas about applying the material there. This is also the section of the journal for your notes from other resources, such books or articles you read or conversations with colleagues.

Resources available to you

Although these modules can be completed without referring to additional resource materials, your experience and that of your students can be enriched if you use other resources as well. There is a list of resource materials for each module provided at the end of that module.

Preliminary knowledge required for this Mathematics Programme

If it has been some time since you reviewed these principles, you may want to check your background knowledge and study these concepts by reviewing the appropriate modules in Science and Mathematics.

Before you begin, you should ensure that you are sufficiently familiar with the basic concepts of mathematics:

- fractions, especially the manipulation of proper fractions
- basic algebraic equations
- general skill with problem solving, graphing, and relating mathematics to everyday situations—a "maths sense"

ICONS

Throughout each module, you will find the following icons or symbols that alert you to a change in activity within the module.

Read the following explanations to discover what each icon prompts you to do.

	Introduction	Rationale or overview for this part of the course.
	Learning Objectives	What you should be able to do after completing this module or unit.
	Text or Reading Material	Course content for you to study.
	Important—Take Note!	Something to study carefully.
	Self-Marking Exercise	An exercise to demonstrate your own grasp of the content.
	Individual Activity	An exercise or project for you to try by yourself and demonstrate your own grasp of the content.
	Classroom Activity	An exercise or project for you to do with or assign to your students.
	Reflection	A question or project for yourself—for deeper understanding of this concept, or of your use of it when teaching.
	Summary	
	Unit or Module Assignment	Exercise to assess your understanding of all the unit or module topics.
	Suggested Answers to Activities	
	Time	Suggested hours to allow for completing a unit or any learning task.
	Glossary	Definitions of terms used in this module.

CONTENTS

Module 1: Number systems

Module 1
Number systems

Introduction to the module

"God made the whole numbers, all else is the work of man" (Kronecker).

What Kronecker is trying to express in this statement is that properties of numbers can be derived from the basic properties of the whole numbers 0, 1, 2, 3, ...

This module looks at the structure of the number system and different classes of numbers that can be identified within the system. The emphasis is on clearly understanding the concepts as a teacher, and on pupil-centred teaching methods to create a class environment in which pupils can learn the concepts at their level.

Aim of the module

The module aims at:

(a) having you reflect on your present methods in the teaching of numbers in the number system

(b) enhancing your content knowledge so that you may set activities with more confidence on the topic to your pupils

(c) making your teaching of number more effective by using a pupil centred approach and methods such as group discussion, games and investigations

Structure of the module

In Unit 1 you will learn about number sense and how to enhance number sense in pupils. You will distinguish between numbers and numerals and look at powers. In Unit 2 you will learn about the technique of using differences for predicting more terms in a sequence and for finding an expression for the general term in the sequence. Unit 3 looks at polygonal numbers and uses an investigative method to discover multiple relationships among these numbers. In Unit 4 you will learn more about the irrational numbers. In all units you will be required to try out activities with your pupils.

Objectives of the module

When you have completed this module you should be able to create a learning environment for your pupils to enhance their:

(a) number sense

(b) understanding of classes of numbers (squares, cubes, even, odd, primes, factors, multiples, polygonal numbers)

(c) understanding and use of the technique of using difference to analyse sequences

(d) understanding of rational and irrational numbers

by making use of a pupil-centred approach in general and using games, group discussion and investigations to enhance pupils' learning.

Unit 1: Number sense, numerals and powers

Introduction to Unit 1

Pupils frequently can manipulate numbers using algorithms without a real awareness of what these numbers represent, their relative size and multiple relationships. In this unit you first look at what number sense is and how to enhance number sense in your pupils. You will study some classes of whole numbers such as powers, factors, prime numbers and multiples.

Purpose of Unit 1

The aim of this unit is to widen your knowledge and to reflect on and try out activities in the classroom that allow your pupils to enhance their feel for numbers and number operations and their understanding of powers, factors, prime numbers and multiples.

Objectives

When you have completed this unit you should be able to:

- state and explain the general objective for pupils who are learning about numbers and their relationships
- develop and use activities to enhance number sense in pupils
- justify using activity-based learning methods for learning of mathematics
- distinguish between a number and a numeral
- develop and use investigative activities with numbers in the classroom
- represent even numbers, odd numbers, consecutive numbers, consecutive odd and even numbers, square numbers, cubes and multiples of a whole number in algebraic form
- illustrate, give examples of and explain power, base, index, factors, multiples
- give examples of and explain prime numbers
- illustrate, give examples of and explain rectangular numbers and oblong numbers
- use algebra to prove relationships among whole numbers

Time

To study this unit will take you about four hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topics.

Unit 1: Number sense, numerals and powers

Section A: A reflection on teaching mathematics

All teachers have their ideas (more or less outspoken) about what the nature of mathematics is (Is it a true, unchangeable body of knowledge? Is it a product of mankind and hence ever changing and expanding?), what the most effective way is to teach mathematics (Telling and demonstrating to pupils? Setting tasks for pupils to work on cooperatively?) and how pupils best learn mathematics (Listening to the teacher and following the examples? Working in groups on a task and discussing?). It is important for you to reflect on your present classroom practices and ideas as some of the module ideas might vary from what you are doing presently. You will be asked to try out activities in your classroom, evaluate the effectiveness of the activities, and to report on the feelings of the pupils. Cockcroft, in the report *Mathematics Count* stated that in the teaching of mathematics the following activities should take place:

1. Exposition
2. Consolidation
3. Discussion
4. Practical activities
5. Problem solving
6. Investigations

My teaching approach and methods

Write down how you teach the concepts listed in your syllabus under "Number". Make an outline of the lessons and the main activities in each lesson when covering the topic.

Describe the main features of your lessons and justify their inclusion.

What do you consider the most effective way of teaching the concepts under "Number". Justify your choice.

Make a list of the activities that take place in your lessons on "Number".

Are the Cockcroft six well covered? Justify.

Keep your reflection as you will need it for your final assignment on this module. Now continue to work through this module.

Section B: The general aim of learning about numbers

The aim of number activities is to develop a **number sense** in the pupils.

When asked to find mentally the cost of 64 items costing P0.50 each, one student recalled that 0.50 = $\frac{1}{2}$*, and took half of 64 and concluded that the cost is P32.- .*

Another student mentally multiplied 5 × 64, taking 5 times 4 remembering the 0 and 'carrying' the 2. Then took 5 × 6, getting 30 and added the 2 he carried, giving him 32. Taking the place of the point into account he gave as his correct response P32.- .

A third child gave an answer P32 and explained that 10 items cost P5 and 60 items therefore cost 6 × P5 = P30. The remaining 4 items cost P2. The total for 64 items is then P30 + P2 = P32.

The first and third pupil used their knowledge about number relationships efficiently, the second applied mentally the standard algorithm for multiplication which was not very convenient to do mentally.

In calculating the area of a rectangular room in a house measuring 3 m by 4.5 m a student gave the answer 144 m^2.

Looking at the answer and being aware of possible areas of rooms the student should have realized that the answer is unreasonably big. Using approximations to estimate the answer the student should have realized that the answer should be in the order of $3\ m \times 5\ m = 15\ m^2$.

7.5% interest was paid on an investment of P250.- Using a calculator a student gave P 187.5 as the answer for the interest received.

This student failed to enter the figures correctly (ignoring the point and using 75% instead of 7.5%) and never questioned the display on the calculator.

Similar situations as described above are probably familiar to you as a teacher. Apart from the first and third pupil in the first example, the others did not show 'number sense'. As teachers we are to aim at developing number sense in our pupils with the emphasis on the processes and strategies used in solving problems rather than on mechanical applications of algorithms and rules to get "the right" answer.

Number sense is characterized by

(1) understanding, representing and using numbers in a variety of equivalent forms in real-life and mathematical problem situations.

(2) having an awareness of multiple relationships among numbers.

(3) recognizing the relative magnitude of numbers.

(4) knowing the relative effect of operating on numbers.

(5) possessing referents for numbers—especially small and large numbers—in the environment. This means having in mind situations one can relate numbers of various magnitude to; for example, knowing that at

assembly in the morning the number of pupils is about 650, that the height of a door is about 2 m, etc.

Self mark exercise 1

1. One-quarter can be represented in many different ways. Write down some ways to represent one-quarter.
2. Write down at least ten questions involving whole numbers that all have an answer of 36.
3. Is $\frac{2}{5}$ greater or smaller than $\frac{1}{2}$? How did you find out quickly?
4. Dividing a number N by another number will always result in a quotient Q smaller than the number N. True or false? Justify your answer.
5. To count from 1 to 1 000 000 (a million) reciting one number per second will take you not more than an hour. True or false? Justify your answer.
6. Explain how each of the above questions relate to one or more of the characteristics of number sense.

Check your answers at the end of this unit.

Section C: Activities to try in the classroom to enhance number sense in pupils

There is overwhelming research evidence that pupils come to a better and lasting understanding of mathematics when they are actively involved in activities that allow group discussion. Discussion in small groups enhances concept building and understanding. Knowledge is, first and foremost, the subjective knowledge of the learner. The learner initially tries to make sense of the new information in the context of the knowledge, experience and beliefs he or she already has. This personal constructed knowledge is expressed and compared with the knowledge of others to come to shared and agreed knowledge.

The activities for class use suggested below are directed to you as a teacher. They are not a complete lesson plan, but are some ideas for developing number sense in your pupils. You can use the ideas and extend them for use in the classroom. The main characteristic of the suggested activities is that they promote understanding and generate rich discussion among the pupils in small groups and between the pupils and the teacher. Before taking an activity to the classroom, work through the activity yourself.

Pupils activity 1: Establishing referent objects for whole numbers

To the teacher:

Pupils are fascinated by large and small numbers. For large numbers pupils need real life situations as referents, i.e., something they can refer to, or relate the number to. Knowing the number of pupils in their school might help them to estimate the number of people attending a football match. The object of this activity is to see how many pupils are aware of the size of some commonly used referents of real life situations. It will also lead to discussion on strategies that can be used to find approximate answers to the questions.

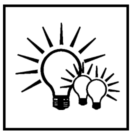
Present the following questions to pupils working together in groups of four. Allow discussion to come to an agreed estimate from the group. Write all the estimates on the board and allow research to find the value that answers each question. To extend the activity, encourage the pupils to come up with additional questions.

To develop a feeling for “How big is a million?” a school project to collect one million beer/soft drink tins could be started.

Questions set to pupils:

1. What is the population of the world?
2. What is the population of your country?
3. What is the population of your town/village?
4. What is the number of pupils in the your school?
5. What is the Government’s budget this year?
6. How many children in the age 12 - 14 are there in the country?

7. How many pupils completed Junior Secondary School last year?
8. How many kilograms of rice are to be cooked in the school to feed all pupils?
9. How far is the Earth from the Sun?
10. What is the diameter of a hair on your head?
11. Can one million Pula in P 10 notes fit in a normal suitcase?
12. How many grains of rice are there in a 1 kg packet?



Pupils activity 2: Using context to determine reasonable values

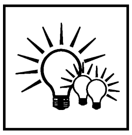
To the teacher:

Pupils working in groups of four, should discuss and insert the missing values in the activity below. As an extension ask pupils in one group to write a story with numerical data structured as the example below. Another group can then place the appropriate values in the place in the story. A newspaper article or text in a book with lots of numerical data, blanking out the values and writing them above the article for pupils to insert at the appropriate place could be used as another activity.

Ask pupils to complete this newspaper story by inserting at the appropriate place 180, hundreds, fourth, 24th, 400, six, 526, 616 and seventeen into the following story:

PTA Meeting Attracts ________ Participants

An audience of about ________ attended the end of year Chorus organised by the PTA on Monday, November ______. The parents, teachers, and friends heard _____ songs and saw _____ musical skits. After the entertainment, last year's school-attendance award was presented to Opela. She attended all _______ days of the last school year. It was her ______ year of perfect attendance. Then the parents and pupils had snacks. ______ fat cakes were eaten and _____ glasses of soft drinks consumed.



Pupils activity 3: Verifying mathematical statements

To the teacher:

Pupils are to learn to consider whether or not published data are reasonable. Obtain some facts from a published source. Pupils in their groups are to verify the accuracy of the statement based on the given fact. Give different questions to each group. The groups are to present their findings in a report to the class.

FACT: Driving on a dust road stirs up 81 000 tonnes of dust.
If you cover a football field with this dust the layer will be about 16 m deep.

FACT: The world population is about 6 billion (6 000 000 000).
If all the people line up shoulder to shoulder along the equator, the line will go 75 000 times around the world.

FACT: Every day about 2 million tennis balls are rejected after use.
You will need 100 classrooms to pack in all these tennis balls.

FACT: A spaceship is launched to Mars and travels at 12 800 km/h.
It will cover the 80 million km journey in about 6 months.

FACT: The speed of light is 3.0×10^8 m/s.
It will take light from the Sun about 5 minutes to reach the Earth (average distance Sun - Earth is 150 million km).

Unit 1, Assignment 1

1. Choose one of the above three pupils activities and work it out in more detail for use in your class. Take into account the level of understanding of your pupils (differentiate the activity if necessary for the different levels of pupil achievement) and use the local environment as context.

2. Try your activity out in the classroom.

3. Write an evaluation of the lesson in which you presented the activity. Some questions you might want to answer could be: What were the strengths and weaknesses? What aspects of the activity needs improvement? How was the reaction of the pupils? What did you learn as a teacher from the lesson? Could all pupils participate? Were your objective(s) attained? Was the timing correct? What did you find out about pupils' number sense? What further activities are you planning to strengthen pupils' number sense? Were you satisfied with the outcome of the activity? Was the activity different from what you usually do in the class with your pupils? How did it differ?

Present your assignment to your supervisor or study group for discussion.

Section D: Numbers, numerals and powers

The following exercise will help you recall some facts related to numbers.

Self mark exercise 2

1. What is a prime number? List the first five.
2. List the first five even numbers.
3. List the first five odd numbers.
4. What are the first five multiples of 6?
5. List the factors of 24.
6. On the dotted line place ALL the possible correct statements choosing from

is a factor of	is a multiple of
divides	is divisible by
is not a factor of	is not a multiple of
does not divide	is not divisible by

a. 3 12

b. 24 4

c. 2 36

d. 72 18

e. 5 7

f. 6 42

g. 121 11

h. 16 4

7. List the first 6 square numbers.
8. How many factors have a square number? Try 16, 25, and 36.
9. List the first 5 cube numbers.
10. What is meant by HCF and LCM? Illustrate these concepts using the numbers 24 and 56.

Check your answers at the end of this unit.

Numbers and Numerals

As a teacher you should be aware of the difference between a **number** and the representation of that number called the **numeral.** A number is an abstract concept. You can have four cows, four children playing together, four bags of cement. All these examples have in common the idea of ‘fourness’: there are four elements in each set. A numeral is any symbol used to represent that concept. The concept “four” can be represented by numerals

4, IV, $\sqrt{16}$, 3.99999.... , ////, 400% and many others. You may notice that one concept in mathematics may have several representations. It is important to present pupils with multiple representations of a concept so that they may get the concept clear.

Algebraic format of numbers

In proofs and justification it is often useful to represent numbers in algebraic format.

The even whole numbers are 2, 4, 6, 8, ... or in general $2n$ for any natural value of n.

If $2n$ is even then the next number $2n + 1$ will be odd.

Consecutive numbers are numbers following each other in a sequence. For example $2n$ and $2n + 2$ are consecutive even numbers, and so are $2n - 2$ and $2n$; consecutive odd numbers are $2n - 1$ and $2n + 1$, or $2n + 1$ and $2n + 3$.

Three consecutive numbers can be represented by $n - 1$, n, $n + 1$ or by n, $n + 1$, $n + 2$ or by $n - 2$, $n - 1$, n, or by $3n$, $3n + 1$, $3n + 2$ etc. The representation used generally does not matter although taking $n - 1$, n, $n + 1$ often leads to simpler algebra. The context of the problem might suggest a particular representation.

The **multiples** of three 3, 6, 9, 12, 15, ... can in general be represented by $3n$ for natural n.

$3n - 1$, $3n$, $3n + 1$ are three consecutive numbers. So are $3n$, $3n + 1$ and $3n + 2$. This last format shows also that every natural number is a threefold, a threefold plus 1, leaving remainder 1 when divided by three or a threefold plus 2, leaving a remainder 2 when divided by three.

Using algebra to prove statements about whole numbers

(i) Suppose we want to verify the statement:

The sum of the squares of five consecutive numbers is divisible by 5.

We could proceed as follows:

The numbers $n - 2$, $n - 1$, n, $n + 1$ and $n + 2$ are five consecutive numbers for any natural value of n greater than or equal to three.

The sum of their squares is

$$(n - 2)^2 + (n - 1)^2 + n^2 + (n + 1)^2 + (n + 2)^2$$
$$= n^2 - 4n + 4 + n^2 - 2n + 1 + n^2 + n^2 + 2n + 1 + n^2 + 4n + 4$$
$$= 5n^2 + 10 = 5(n^2 + 2).$$

The expression, sum of the squares of five consecutive numbers which is equal to $5(n^2 + 2)$, has a factor 5 and is hence divisible by 5.

If you would have taken n, $n + 1$, $n + 2$, $n + 3$, $n + 4$ to represent five consecutive numbers, the same result would have emerged:

The sum of their squares is

$$n^2 + (n + 1)^2 + (n + 2)^2 + (n + 3)^2 + (n + 4)^2$$
$$= n^2 + n^2 + 2n + 1 + n^2 + 4n + 4 + n^2 + 6n + 9 + n^2 + 8n + 16$$
$$= 5n^2 + 20n + 30 = 5(n^2 + 4n + 6).$$

The expression has again a factor 5 and hence the sum of the squares of five consecutive numbers is divisible by 5.

(ii) To prove that the sum of three consecutive even numbers is a multiple of 6 you could work as follows:

Let the three even consecutive be $2n - 2$, $2n$ and $2n + 2$ ($n > 1$).

The sum is $2n - 2 + 2n + 2n + 2 = 6n$, which is clearly a multiple of six.

Powers and indices

2^4, 4^2, 5^7, 15^3 are **powers**. Powers are a short way of writing down a repeated multiplication of the same factor.

$2^4 = 2 \times 2 \times 2 \times 2$, the product of 4 factors of 2.

$4^2 = 4 \times 4$, the product of 2 factors of 4.

$5^7 = 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5$, the product of 7 factors of 5.

n^p is the product of p values of n.

In 2^4, 4 is called the **index** of the power (plural, 'indices') and 2 is called the **base.**

2 is the same as 2^1, although the index 1 is not written in final expressions, but sometimes used in computations to help as a reminder.

The index shows in the above cases how many times the base number appears as a factor in the product.

We restrict ourselves here to powers with both n and p being natural numbers, although n could be any real number and the same statement would still apply provided p is a natural number. For example:

$(1.2)^3 = 1.2 \times 1.2 \times 1.2$, the product of 3 factors of 1.2

$(^-5\frac{1}{3})^4 = (^-5\frac{1}{3}) \times (^-5\frac{1}{3}) \times (^-5\frac{1}{3}) \times (^-5\frac{1}{3})$, the product of 4 factors of $(^-5\frac{1}{3})$

Factors of natural numbers

Problem 1

A group of 18 people went to a dinner party. The people in the party were to be seated at tables each containing the same number of people. How many arrangements are possible?

How many factors has 18?

Did you find six possible arrangements and six factors?

The six possible arrangements are

1 table with 18 persons
2 tables with 9 persons
3 tables with 6 persons
6 tables with 3 persons
9 tables with 2 persons
18 tables with 1 person

The six factors of 18 are: 1, 2, 3, 6, 9 and 18. The factors occur in pairs:
1 and 18, 2 and 9, 3 and 6 as

$18 = 1 \times 18 = 2 \times 9 = 3 \times 6 = 6 \times 3 = 9 \times 2 = 18 \times 1$.

Note that 1×18 and 18×1 are not the same in meaning (1 table with 18 persons is not the same as 18 tables with 1 person each) although they have the same value.

Similarly 2×9 and 9×2 have different meanings, but the same value.
$2 \times 9 = 9 + 9$ is the sum of two terms of nines and
$9 \times 2 = 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2$ is the sum of nine terms of twos.

Problem 2

a. Copy and complete a table for the numbers 1 to 100

Number	Factors	Number of factors
1	1	1
2	1, 2	2
3	1, 3	2
4	1, 2, 4	3
5		
6		
....		

b. List the numbers with exactly 2 factors. What name is given to these numbers?

c. List all the numbers with exactly 3, 5, 7, ... factors. What is the name of these numbers?

d. Represent the numbers in c algebraically.

Check your answers at the end of this unit.

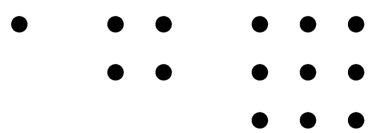
Primes, squares and rectangular numbers

The numbers with exactly 2 factors (1 and the number itself) are called **prime numbers**.

2, 3, 5, 7, 11, 13, 17, 19, ... are prime numbers.

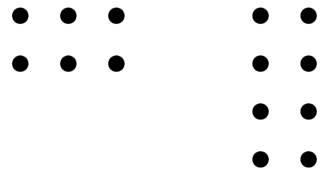
There is no pattern in this sequence, i.e., there is no expression that can generate all the prime numbers.

The numbers 1, 4, 9, 16, 25, ... with an odd number of factors (1, 3, 5, 7, ...) are called **square numbers.** If represented using a dot pattern, the dots can be placed to represent a square. Each square number is the result of multiplying a natural number by itself—1×1, 2×2, 3×3, 4×4, ...—which can also be written as 1^2, 2^2 , 3^2, 4^2, ...



1 4 9

Numbers with three or more factors are called **rectangular numbers** as, when using dots, they can form a rectangular pattern. The diagram below illustrates the rectangular numbers 6 and 8. The pattern is always to have at least two rows (or columns).



6 8

Self mark exercise 3

1. Illustrate, using a dot pattern, that 12 is a rectangular number. How many different patterns can you make?
2. Is a square number a rectangular number? Justify your answer.
3. Explain why the following dot pattern is not a correct way to represent the rectangular number 6.

6

4. a. Explain that of any three consecutive numbers, one of the three must be a threefold, i.e., a multiple of three.

   b. Explain that of any four consecutive numbers, one must be a fourfold, i.e., a multiple of four.
5. Justify the following statements using examples and give an algebraic proof.

   a. The product of three consecutive odd numbers is divisible by 3.

   b. The sum of three consecutive odd numbers is divisible by 3.

   c. The product of two consecutive even numbers is always the difference between a square number and 1.

   d. The square of any whole number is either a multiple of three or has a remainder 1 when divided by three.

   e. The square of any odd number is one more than a multiple of 4.

   f. The difference between two consecutive square numbers is an odd number.

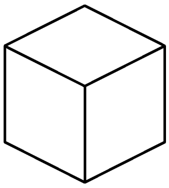
   g. If the square of a whole number is even then the whole number must be even.

Check your answers at the end of this unit.

Rectangles that are not squares are called **oblongs**. The rectangular numbers could be placed into two groups: **square numbers** and **oblong numbers**.

Cube numbers

The numbers $1^3 = 1$, $2^3 = 8$, $3^3 = 27$, $4^3 = 64$, $5^3 = 125$, ... are called cube numbers because they can be represented using unit cubes (cubes with an edge of length 1 unit) to build cubes as illustrated.



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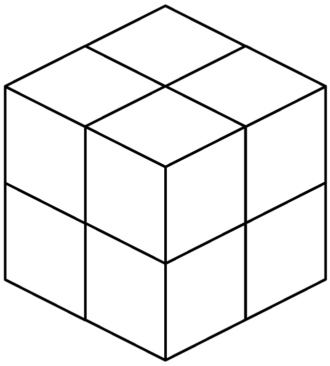

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Self mark exercise 4

1. Illustrate using (i) square grid paper (ii) isometric paper that 27 is a cube number.
2. Is a cuboid a cube or a cube a cuboid? Justify your answer.
3. How would you define cuboidal numbers? Illustrate some cuboidal numbers using square grid paper and isometric paper.
4. The number 60 is a cuboidal number. In how many different ways could you illustrate this?
5. a. A wooden 3 cm × 3 cm × 3 cm cube is painted blue on the outside and cut into 27 unit cubes (1 cm × 1 cm × 1 cm). How many of the unit cubes will have (i) no face painted (ii) 1 face painted (iii) two faces painted (iv) three faces painted?

   b. What if the original cube measured 4 cm × 4 cm × 4 cm or 5 cm × 5 cm × 5 cm or ... was painted at the outside and cut into unit cubes?
   Tabulating your results might be helpful.

Dimensions	Total number of unit cubes	Number of unit cubes with... faces painted blue 0	1	2	3
$2 \times 2 \times 2$	8	0	0	0	8
$3 \times 3 \times 3$	27	..	..	..	..

   c. Can you generalize for an n cm × n cm × n cm cube?

   d. What are the dimensions of a cube with 3750 unit cubes painted at one face?

   e. What are the dimensions of a cube with 1440 unit cubes painted on two faces?

   f. What are the dimensions of a cube with 64 000 unpainted unit cubes?

Check your answers at the end of this unit.

Investigating patterns in powers

Problem 3

The last digit (unit digit) of the powers of the natural numbers follow patterns. In this investigation you are to find and describe those patterns.

1. Copy and complete the ‘patterns in the powers chart’ table partly illustrated below.

Patterns in the powers chart

N^1	N^2	N^3	N^4	N^5	N^6	..	..	Units digits
1	1	1	1	1	1			1
2	4	8	16	32	64			2, 4, 8, 6
3								
4								
5								
6								
7								
8								
9								
10								

2. a. Look at the rows: what are the last digits of powers of 2, 3, ... etc.?

   b. In what cycle are the last digits of the powers of each number repeating?

   c. How are the indices of the powers of the numbers related to the last digit?

3. Use the pattern to find the last digit of 2^{1999}, 3^{1999}, ...10^{1999} and 1999^{1999}.

4. Look at the columns: what are the possible last digits of square numbers? cube numbers? n^4?

5. Use your pattern to decide whether the number 345 678 965 232 can be a square number, a cube, a fourth power, etc.

6. Justify the following statements (as a challenge: try to prove them deductively).

   a. All cubes are multiples of four, or of four added or subtracted 1 (i.e., one more or less than a multiple of four).

   b. All cubes are multiples of seven, or of seven added or subtracted 1.

7. Set some questions of your own on powers and investigate them.

Check your answers at the end of this unit.

Teacher's notes on Pattern in the powers chart

You just investigated patterns in the chart of powers of the natural numbers. The investigation can be presented to pupils, for example starting as a challenge.

Pupils activity 4

What is the last digit of 2^{10}?

What of 2^{100}?

Be sure pupils understand the question. *Expand* 2^{10}. *What is the digit in the unit place?* (Use calculator to obtain the answer.)

[2^{10} = 1024 so the digit in the unit place is 4].

Allow pupils time to explore the problem. Trying the calculator for 2^{100} will not work as the calculator will give an approximate result in scientific notation (1.2676506×10^{30}). Suggest looking at smaller exponents to find a pattern.

The unit digits repeat in a cycle of four: 2, 4, 8, 6, 2, 4, 8, 6, 2, 4, 8, 6, 2, ... Pupils are to observe where each power of 2 occurs in the cycle: 2 corresponds to the exponents 1, 5, 9, ... or for the exponents that have remainder 1 when divided by 4 (or are 1 more than a multiple of 4). Similarly 4 occurs for the exponents 2, 6, 10, ... or for the exponents that have a remainder 2 on division by 4; 8 will occur for the exponents that have a remainder 3 when divided by 4 and 6 will occur for exponents that have a remainder 0 (or are multiples of four) when divided by 4.

So 2^{100} will end in 6, 2^{103} ends in 8, etc.

Extension

The original question, restricted to the powers of 2, can now be extended for powers of all whole numbers 1 through 10. Let pupils investigate these powers. An amazing pattern emerges. The longest cycle for the units digit is four.

Tabulating the results lead to other patterns to be considered: endings of square numbers [1, 4, 9, 6, 5, 6, 9, 4, 1, 0 i.e., a square number can never end in 2, 3, 7 or 8], cube numbers [all digits 0 to 9 can occur as units].

Which powers are the same? (For example $2^4 = 4^2 = 16$; $2^6 = 4^3 = 64$ Why? This can lead to the generalization $[2^n]^m = 2^{nm}$).

The other exponential rules can be investigated from the table. Also questions such as:

Does $2^3 \times 3^2 = 6^6$? {No} Why ?
Does $2^3 \times 2^2 = 2^6$? {No} Why ?
Does $2^3 \times 3^3 = 6^3$? {Yes} Why ?
can be explored.

Patterns in the powers chart

N^1	N^2	N^3	N^4	N^5	N^6	..	..	Units digits
1	1	1	1	1	1			1
2	4	8	16	32	64			2, 4, 8, 6
3	9	27	81	243	729			3, 9, 7, 1
4	16	64	256	1024	4096			4, 6
5	25	125	625	3125	15625			5
6	36	216	1296	7776	46656			6
7	49	343	2401	16807	117649			7, 9, 3, 1
8	64	512	4096	32768	262144			8, 4, 2, 6
9	81	729	6561	59049	531442			9, 1
10	100	1000	10 000	100 000	1 000 000			0

Some further questions for pupil investigation:

What is the digit in the units place for 17^{100}, 25^{50}, 31^{1000}?

Investigate the **last two digits** of powers of numbers.

You have completed this section and should be able to answer the following questions.

Self mark exercise 5

1. Give two different meanings of the word (i) ‘square’ and (ii) ‘cube’.
2. Explain why square numbers have always an odd number of factors while all other numbers have an even number of factors.
3. Explain the different meaning of 3×6 and 6×3.
4. Give five different representations (numerals) of ‘tenness’.
5. Is $n^2 + n + 41$ a prime number for all natural values of n? Investigate.
6. A pupil wrote $2^3 = 6$ and $3^3 = 9$. What error did the pupil make? Which remedial steps would you take to help the pupil to overcome the problem?
7. Explain why 1 is NOT a prime number, while 2 is.
8. Give a dot illustration of a oblong number and a square number.
9. How many factors has the rectangular number 120? In how many different ways could you make a dot pattern to illustrate that 120 is a rectangular number?
10. How many factors has the rectangular number 121? In how many different ways could you make a dot pattern to illustrate that 121 is a rectangular number?
11. Comparing the rectangular numbers 120 and 121, describe their difference.
12. Explain, using examples, the meaning of power, base and index.
13. Is 225 736 192 672 a square number? How can you quickly tell?

Check your answers at the end of this unit.

Unit 1: Practice activity

1. Choose one of the previous three problem solving activities and work it out in more detail for use in your class as a group task. Take into account the level of understanding of your pupils (differentiate the activity if necessary for the different levels of pupils' achievement).
2. Try your activity out in the classroom with a form 2 or 3. Pupils should work together in groups of 4 and come up with ONE agreed response.
3. Write an evaluation of the lesson in which you presented the activity. Some questions you might want to answer could be: What was / were the objectives of your lesson? How did you structure the activity set to the groups and why? What was the criteria you used to form the groups? What were the strengths and weaknesses of the activity? How was the reaction of the pupils? Did they like the activity? Did they like to work as a group? Were there any problems you have to take into account when using a similar method? What did you learn as a teacher from the lesson? Could all pupils participate? Were your objective(s) attained? Was the timing correct? What did you find out about pupils' number sense? What further activities are you planning to strengthen pupils' number sense? Were you satisfied with the outcome of the activity? Were the objectives achieved? How do you know? Was the activity different from what you usually do in the class with your pupils? How did it differ?

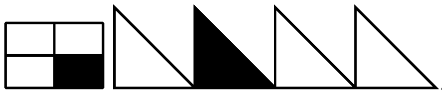
Present your assignment to your supervisor or study group for discussion.

Summary

Unit 1 discussed the challenges that one faces when trying to *teach* mathematics in a way that causes students to *acquire* mathematics skills (rather than the simpler skills of moving symbols correctly on the page). It presented examples of teaching through pictures, diagrams and everyday objects, as well as through math symbols. The overall aim—which should also be your overall aim in your classroom—was to show students that the manipulations they learn to do with numbers correspond to the way objects in the world also behave.

Unit 1: Answers to self mark exercises

Self mark exercise 1

1.  , $\frac{1}{4}$, $\frac{2}{8}$, and many others.

2. The emphasis is on <u>examples in context</u>. Number, calculations are to relate to something to make sense to pupils. E.g.

   a) What is the cost of 9 bars of chocolate at P4 each?

   b) In a youth group there are an equal number of boys and girls. If there are 18 girls in the group, how many are there in the group altogether?

   c) A rectangular card measures 12 cm by 3 cm. What is its area?

   d) Tiles measure 9 cm by 4 cm. What is the length of the side of the smallest square that can be covered by these tiles without having to cut any?

3. $\frac{2}{5} < \frac{1}{2}$. Convert to decimals or percent, or fractions with common denominator ($\frac{4}{10}$, $\frac{5}{10}$).

4. Divide $^-4$ by 2, the quotient $^-2$ is greater than $^-4$.

   divide 4 by 0.5, the quotient 8 is greater than 4, Hence dividing *n* by a number might give as a quotient a result greater than *n*.

5. It takes 1 000 000 seconds which is about 277.8 hours (1 dp) or 11.6 days (1 dp).

6. Question 1 relates to (1) & (2); Question 2 relates to (1); Question 3 to (3); Question 4 to (4) and question 5 to (3) & (5).

Self mark exercise 2

1. Prime numbers are numbers with exactly two factors (1 and the number itself)

   2, 3, 5, 7, 11, ..

2. 2, 4, 6, 8, 10, ..

3. 1, 3, 5, 7, 9, ..

4. 6, 12, 18, 24, 30, ..

5. 1, 2, 3, 4, 6, 8, 12, 24

6. (a)(c)(f) is a factor of, divides, is not divisible by, is not a multiple of
   (b)(d)(g)(h) is a multiple of, is divisible by, is not a factor of, does not divide
   (e) is not a factor of, does not divide, is not divisible by, is not a multiple of

7. 1, 4, 9, 16, 25, 36, ..

8. an odd number (3, 5, 7, ..) but never even number of factors

9. 1, 8, 27, 64, 125, ..

10. HCF highest common factor of 24 and 56 is 8
    LCM lowest common multiple of 24 and 56 is 168, i.e., first number being multiple of both

Problem 2

1b. prime number have 2 factors: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 87, 89, 97

c. 1 factor has only 1
   Numbers with odd number of factors are square numbers.
   3 factors have 4, 9, 25, 49
   5 factors have 16, 81
   7 factors has 64
   9 factors have 36, 100
   Numbers with 4, 6, 8, ..or more even number of factors are oblong numbers.
   4 factors have 6, 8, 10, 14, 15, 21, 22, 26, 27, 33, 34, 35, 38, 39, 46, 51, 55, 57, 58, 62, 65, 69, 74,, 77, 82, 85, 86, 91, 93, 94, 95
   6 factors have 12, 18, 28, 32, 44, 45, 50, 52, 63, 68, 75, 76, 98, 99
   8 factors have 24, 30, 40, 42, 54, 56, 66, 70, 78, 88
   10 factors have 48, 80, 84
   12 factors have 72, 90, 96

d. Square numbers can be represented algebraically as n^2, where n is a whole number.

Self mark exercise 3

1. $12 = 2 \times 6 = 3 \times 4 = 4 \times 3 = 6 \times 2$. Four different patterns if you consider 3×4 and 4×3 as different patterns, 2×6 and 6×2.

2. Except for 1, all square numbers are rectangular as they have at least 3 factors.

3. By convention a rectangular number dot pattern needs at least 2 rows (columns) and NOT one row (or column).

4a. If you divide a number by three the remainder is either 0, 1 or 2.

If n, $n + 1$, $n + 2$ are three consecutive numbers and you divide n by three then
(i) if remainder 0 then n is multiple of 3 (ii) if remainder 1 then $n + 2$ will be multiple of 3, if remainder 2 then $n + 1$ must be multiple of 3. Hence one of the three numbers is multiple of 3.

4b. Same reasoning as in 4a. If n, $n + 1$, $n + 2$, $n + 3$ are four consecutive numbers then when dividing n by 4 remainder is 0, 1, 2 or 3. If remainder 0, n is fourfold, if remainder 1 then $n + 3$ is fourfold. Complete the argument.

5a. Taking the consecutive numbers $2n$, $2n + 1$, $2n + 2$, $2n + 3$, $2n + 4$, $2n + 5$, the underlined ones represent three consecutive odd numbers. Use similar reasoning as in question 4. If $2n + 1$ is divided by 3 remainder is 0, 1 or 2. If 0 then $2n + 1$ is a threefold, if remainder 1 then $2n + 3$ will be threefold and if remainder 2 then $2n + 5$ is a threefold. Hence one of the three consecutive odd numbers is a threefold and hence their product.
$(2n + 1)(2n + 3)(2n + 5)$ must have a factor 3 i.e., is a threefold.

5b. Three consecutive odd numbers can be represented by $2n – 1$, $2n + 1$, $2n + 3$; their sum is $6n + 3 = 3(2n + 1)$, i.e., is a threefold.

5c. $2n(2n + 2) = 4n^2 + 4n = (2n + 2)^2 – 4 = (2n + 2)^2 – 2^2 = 4[(n + 1)^2 – 1]$
This is the difference of the square of a number and 1.
E.g., $4 \times 6 = 24 = 5^2 – 1^2$, $6 \times 8 = 48 = 7^2 – 1$

5d. $n^2 = n \times n$. Remember from question 4 that n is either a threefold, a threefold + 1 or a threefold + 2.

If $n = 3m$ (a threefold) $n^2 = 3m \times 3m$ which is clearly a threefold.

If $n = 3m + 1$ (a threefold + 1), $n^2 = (3m + 1)(3m + 1) = 9m^2 + 6m + 1 = 3[3m^2 + 2m] + 1$, which when divided by 3 gives remainder 1.

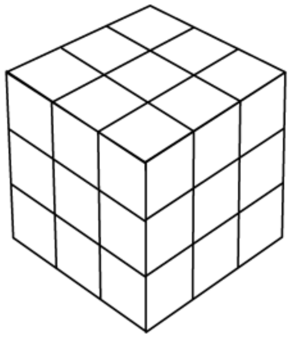
If $n = 3m = 2$ (a threefold + 2),
$n^2 = (3m + 2)(3m + 2) = 9 m^2 + 12 m + 4 = 3(3m^2 + 4m + 1) + 1$, which gives again remainder 1 when divided by 3.

5e. $(2n + 1)^2 = 4n^2 + 4n + 1 = 4[n^2 + n] + 1$, which is a fourfold + 1.

5f. $(n + 1)^2 – n^2 = 2n + 1$, which is an odd number.

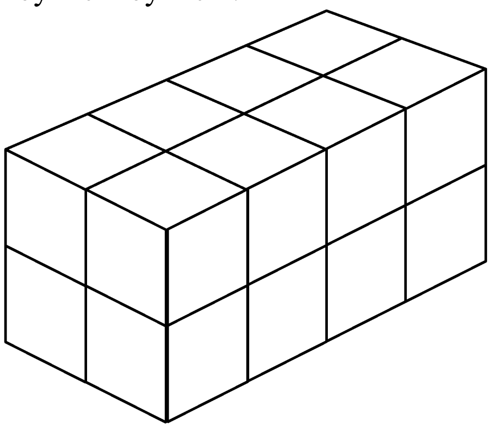
5g. If n^2 is even it must contain a factor 2 and so $n^2 = 2M$, $n \times n = 2M$. But if $n \times n$ has a factor 2 it means n must have a factor 2, i.e., is even.

Self mark exercise 4

1. 

2. A cube is a special cuboid, i.e., a cuboid with all edges equal in length.

3. Cuboid numbers are numbers that can be express as product of three factors (> 1).
   For example $12 = 2 \times 2 \times 4$ which can make a cuboid with length of sides 2 cm by 2 cm by 4 cm.



4. $60 = 3 \times 4 \times 5 = 2 \times 5 \times 6 = 2 \times 3 \times 10 = 2 \times 2 \times 15$.
   This gives 4 different ways taking the combinations $3 \times 4 \times 5$, $3 \times 5 \times 4$, $4 \times 3 \times 5$, $4 \times 5 \times 3$, $5 \times 3 \times 4$, $5 \times 4 \times 3$ as 'the same'.

5a. (i) 1 (ii) 6 (iii) 12 (iv) 8

5b. $4 \times 4 \times 4$: (i) 8 (ii) 24 (iii) 24 (iv) 8 $5 \times 5 \times 5$: (i) 27 (ii) 54 (iii) 36 (iv) 8

5c. $n \times n \times n$:
(i) $(n-2)^3$
(ii) $6(n-2)^2$
(iii) $12(n-2)$
(iv) 8

5d. 27

5e. 122

5f. 42

Problem 3

1. 2. Patterns in the powers chart

N^1	N^2	N^3	N^4	N^5	N^6	N^7	N^8	N^9	Units digits
1	1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512	2, 4, 8, 6
3	9	27	81	243	729	2187	6561	19683	3, 9, 7, 1
4	16	64	256	1064	4256	17024	68096	272384	4, 6
5	25	125	625	3125	15625	78125	390625		5
6	36	216	1296	7776	46656				6
7	49	343	2401	18807	131649				7, 9, 3, 1
8	64	512	4096	32768	262122				8, 4, 2, 6
9	81	729	6461	58149	523341				9, 1
10	100	1000	10 000	100 000	1 000 000				0
Last digits	1, 4, 6, 5, 9, 0	all	6, 1, 5, 0	all	1, 4, 6, 5, 9, 0	all	0, 1, 5, 6	all	

3. Last digit: 8, 7, 4, 5, 6, 3, 2, 9, 0
   1999^{1999} last digit 9
4. See table
5. Number could be power of number to ODD index, e.g., n^3, n^5, n^7, etc.

6a. Any number is fourfold, fourfold + 1, fourfold + 2 or fourfold + 3, i.e., if you divide a natural number by 4 the remainder will be 0, 1, 2 or 3.

Look at the cubes of $4n$, $4n + 1$, $4n + 2$, $4n + 3$.

$(4n)^3 = 64n^3$ which is multiple of 4

$(4n + 1)^3 = 64n^3 + 48n^2 + 12n + 1 = 4[16n^3 + 12n^2 + 3n] + 1= 4M + 1$, fourfold + 1

$(4n + 2)^3 = 64n^3 + 96n^2 + 48n + 8 = 4[...]$, a fourfold

$(4n + 3)^3 = 64n^3 + 144n^2 + 108n + 27 = 4[16n^3 + 36n^2 + 27n + 7] – 1$, fourfold – 1

6b. The first cubes are 1, $8 = 7 + 1$, $27 = 4 \times 7 - 1$, $64 = 9 \times 7 + 1$, $125 = 18 \times 7 - 1$, $216 = 31 \times 7 - 1$, $343 = 49 \times 7$, $512 = 73 \times 7 + 1$, $729 = 104 \times 7 + 1$, ..

The pattern suggests that each cube is a multiple of seven or of seven added or subtracted 1.

For a formal prove, work along the lines as in 6a. Each natural number when divided by 7 leaves remainder 0, 1, 2, 3, 4, 5 or 6, hence can be written as $7m$, $7m + 1$, $7m + 2$, $7m + 3$, $7m + 4$, $7m + 5$ or $7m + 6$.

Work now the cube of each of these numbers. For example:

$(7m + 2)^3 = 7^3m^3 + 3.7^2m^2 + 3^2$.
$7m + 8 = 7[49m^3 + 21m^2 + 9\,m + 1] + 1 = 7M + 1$, a sevenfold added 1.

Work the remaining six possibilities similarly.

7. For example: Investigate the last 2 digits of the power of N.

   Investigate the sum of the digits of the powers of N.

Self mark exercise 5

1. Square - (1) rectangle with equal sides (2) number multiplied by itself $n \times n = n^2$

   Cube (1) cuboid with all edges equal in length (2) product $n \times n \times n = n^3$

2. Factors occur in pairs $6 = 2 \times 3 = 1 \times 6$, $8 = 1 \times 8 = 2 \times 4$, etc. So in general a number will have an even number of factors. However a square number has one set of equal factors, e.g.:

   $9 = 1 \times 9 = 3 \times 3$, $16 = 1 \times 16 = 2 \times 8 = 4 \times 4$, hence the total number of factor will end up being an odd number.

3. $3 \times 6 = 6 + 6 + 6$ $6 \times 3 = 3 + 3 + 3 + 3 + 3 + 3$

4. 10, X, \\\\\ \\\\\, $\sqrt{100}$, ***** *****, $\frac{20}{2}$, 2×5

5. Did you try 41? 82? or other multiples of 41?

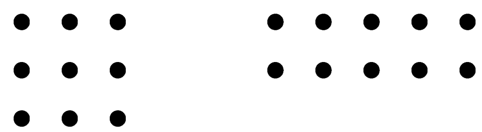
6. Pupil took $2^3 = 2 \times 3$ and $3^3 = 3 \times 3$, confusing raising to a power with multiplication of base and index.

   Go with the pupil through the four steps of remediation:

   (i) ask the pupil to explain how he/she worked the question
   (ii) create mental conflict by asking the pupil to use the calculator to find 2^3, 3^3
   (iii) develop the correct concept by going to the meaning (convention) of $2^3 = 2 \times 2 \times 2$, product of three factors two, 3^3 as product of three factors three, encouraging the pupil to write out the ‘in-between-step’ and to verbalize, e.g., $(2.4)^4 = 2.4 \times 2.4 \times 2.4 \times 2.4$, product of 4 factors 2.4

(iv) set some drill and practice questions to consolidate the correct concept (include fractions / decimals)

7. One (1) has only 1 factor while by definition prime numbers have exactly 2 factors. Two (2) meets this criteria for prime numbers as it has as factors 1 and 2.
8. Below square number 9 and oblong number 10



Square number Oblong number

9. 120 has 16 factors: 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120

   Dot patterns 7 (or 14 if you take 2×60 and 60×2 as different).
   N.B. 1 x120 = 120×1 are by convention NOT taken as representations of rectangular numbers.
10. 121 has as factors 1, 11, 121. Only one dot pattern is possible 11×11.
11. Differences you could mention are

    120 is even, 121 is odd

    120 is a rectangular number (and NOT square), 121 is a square number

    120 has 16 factors (even number), 121 only 3 (odd number)

    120 uses three different digits, 121 only 2
12. In a^n, a is called the base, *n* the index and a^n is called a power.
13. See table in P3; square numbers end in 1, 4, 6, 5, 9 or 0 NEVER in 2.

Unit 2: Sequences

Introduction to Unit 2

In Unit 1, you did some investigations. In most of them a pattern of numbers emerged. In problem solving and investigations, **looking for a pattern** is one of the strategies to consider. Patterns are very common around us. Think of tile patterns, patterns on cloth, patterns used in decoration, patterns in knitting, etc. These patterns can frequently be described using numbers. Number patterns are therefore studied in mathematics and various techniques have been developed to describe in general terms the pattern in a sequence of numbers.

Purpose of Unit 2

The aim of this unit is to familiarize you with:

- the method of differences to analyse sequences, and to find an expression for the *n*th term of a sequence
- the use of games to consolidate pupils' concepts
- the use of problem solving activities as a learning method

Objectives

When you have completed this unit you should be able to:

- express the pattern in a sequence of numbers in words
- use the method of differences to analyse sequences of numbers
- use the method of differences to find a linear or quadratic expression for the general term in a sequence
- give simple examples of sequences with a clear pattern where the expression for the *n*th term is not linear or quadratic
- use problem solving and investigative activities in the classroom, related to number patterns and sequences
- justify the use of games in consolidation of concepts
- evaluate the effectiveness of using games in pupils' learning

Time

To study this unit will take you about 6 hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topic.

Unit 2: Sequences

Section A: Introduction

Patterns occur all around us. The calendar of May 1999 is shown.

S	M	T	W	T	F	S
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30	31.					

Take any square of four numbers. For example the 'eleven' square:

11	12
18	19

$\mathbf{11} + 19 = 12 + 18 = 30 = 2 \times \mathbf{11} + 8$

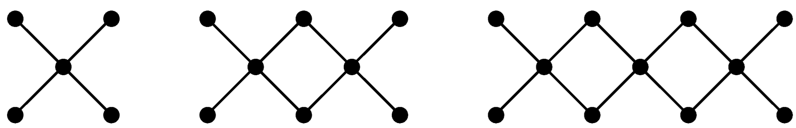
Repeat for other squares. Is the diagonal sum always equal to twice the 'starting' number of the square plus 8?

Here is another number pattern:

$$3 \times 11 = 33$$
$$33 \times 11 = 363$$
$$333 \times 11 = 3663$$
$$3333 \times 11 = 36663$$

How does the pattern continue? Can you explain?

Here you have a 'growing' dot-line pattern:



What would be the next dot-line pattern? Is there a relationship between the number of dot and the number of line segments?

Reflection

Before you study this unit think about the following questions and write down your responses.

- Was a lot of attention paid to patterns and sequences while you were studying?
- Why yes or no? What was covered?
- Do you consider the topic to be important or not? Justify your answer.
- Are you setting activities (and which) to your pupils related to patterns and sequences? Why? or Why not?
- Can you list situations in real life in which patterns / sequences are involved?

While working through this unit refer back to your 'reflection notes'.

Section B: Finding an *n*th term formula

You may have found it hard in the painted cube problem to come up with the general expression for the n cm × n cm × n cm cube. In this section we look at a technique that is frequently very useful when trying to find a general algebraic expression for the nth term in a sequence. It is called the **method of differences**. The method will help you in the next unit when we look at figurative numbers or polygonal numbers. This are numbers that can be illustrated with dot patterns of regular polygons.

Consider the sequence

Sequence	4		8		12		16 ...		4n
(First) difference		4		4		4		4	

The numbers in the sequence are called terms. 4 is the value of the first term t_1. We write $t_1 = 4$. 8 is the value of the second term represented by t_2, i.e., $t_2 = 8$. The next term in the sequence is found by adding four to the previous term. The 'rule' can be expressed as: start with 4 and continue to add four.

The general term in the sequence, the nth term is $t_n = 4n$. Check that for $n = 1, 2, 3, 4, 5, \ldots$ you obtain the terms of the sequence.

Section C: Investigating the *n*th term of a sequence

In this section you are going to study how the pattern in a sequence relates to the expression for the *n*th term.

Problem 1

Study the following sequence

3, 7, 11, 15, 19, ..., $4n - 1$, ... $t_n = 4n - 1$

8, 15, 22, 29, 36, ..., $7n + 1$, ... $t_n = 7n + 1$

30, 25, 20, 15, 10, ... , $^{-}5n + 35$, .. $t_n = {}^{-}5n + 35$

1. a) What are the (first) differences between consecutive terms in the first sequence?

   Answer _____

   b) Express the 'rule' for the sequence in words:

   ____

   c) Where do you find the constant first difference in the rule for the *n*th term of the sequence?

   ____

2. a) What are the differences between consecutive terms in the second sequence? Answer _____

   b) Express the 'rule' for the sequence in words:

   ____

   c) Where do you find the constant first difference in the rule for the *n*th term of the sequence?

   ____

3. a) What are the differences between consecutive terms in the third sequence?

   Answer _____

   b) Express the 'rule' for the sequence in words:

   ____

   c) Where do you find the constant first difference in the rule for the *n*th term of the sequence?

   ____

Check your answers at the end of this unit.

The constant first differences in each sequence we find in our rule as the coefficient of *n*.

In the first sequence, the constant first differences were 4 and the rule for the *n*th term started with $4n$ + ...

Similarly the constant first differences for the second sequence were 7, for the third $^-5$, the *n*th term started with respectively $7n$ ... and ^-5n....

How to find the rule?

Consider the following sequence:

Sequence	75		66		57		48	...
(First) differences		$^-9$		$^-9$		$^-9$		

We expect now the rule to be of the form: ^-9n + (number)

As the first number is 75, taking n = 1 $^-9 \times 1$ + (number) = 75

Solving for (number): (number) = 75 + 9 = 84

Our rule for the nth term in the sequence is: $t_n = {}^-9n + 84$

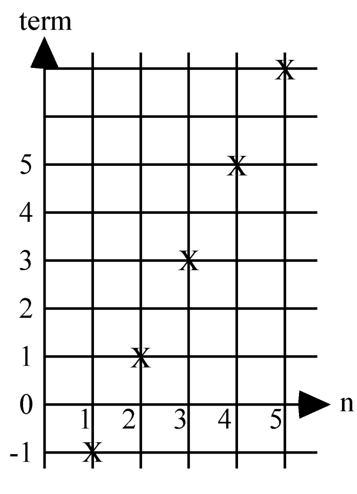
Check that this gives the correct terms: substitute for n the values 1, 2, 3, ...

As our rule is of the form $y = px + q$

(when y is plotted against x it will give us pair of corresponding points that, if joined, would be on a straight line) we say that it is a linear rule.

The graph below illustrates the sequence given by $t_n = 2n - 3$

n	1	2	3	4	5	...
t_n	$^-1$	1	3	5	7	...

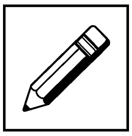


Note that the graph consists of isolated points. Connecting the points would be meaningless as there is no 1.5th or 1.6th term.

If the terms would be connected the points are on the line with equation $y = 2x - 3$, which is a linear equation and hence the sequence is also said to be a sequence with a linear rule.

Section D:
Using the method of differences to find the *n*th term

Looking at differences between consecutive terms is a method that can lead to a formula for the *n*th term of a sequence. You are first going to study linear rules for the *n*th term. Rules that are of the form $t_n = an + b$ $(a \neq 0)$



Self mark exercise 1

Using the method of differences to find the nth term.

1. Find rules that generate the terms of the following sequences.
   Write the rules as t_n = ...

Test the rule by finding the next few terms.

a) 6, 10, 14, 18, ... t_n = ____________

b) 2, 7, 12, 17, ... t_n = ____________

c) 52, 49, 46, 43, ... t_n = ____________

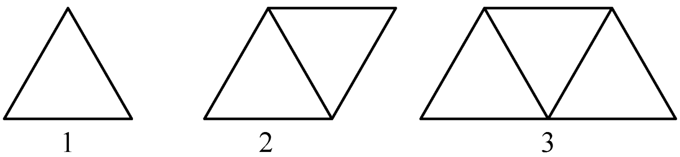
d) $^-5, ^-1, 3, 7$, ... t_n = ____________

e) $^-2, ^-4, ^-6, ^-8$, ... t_n = ____________

f) 2.5, 3, 3.5, 4, 4.5, ... t_n = ____________

g) $6, 5\frac{1}{4}, 4\frac{1}{2}, 2\frac{3}{4}$, ... t_n = ____________

2. a. A sequence of equilateral triangles is made by placing them side by side as illustrated.



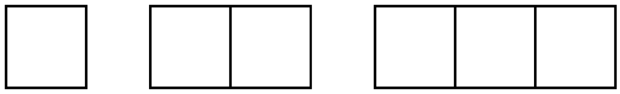
Find the perimeter if the 'train' is one, two, three, four, ... equilateral triangles long. Complete the following table

Number of triangles	Perimeter
1	3
2	
3	
4	
5	
6	
n	

Self mark exercise 1 continued on next page

Self mark exercise 1 continued

2 b. Now consider a "train" using squares.



Find the perimeter if the 'train' is one, two, three, four, ... squares long. Complete the following table.

Number of squares	**Perimeter**
1	4
2	
3	
4	
5	
6	
n	

2 c. Now draw "trains" with regular pentagons, regular hexagons, regular heptagons. Find the perimeter if the 'train' is one, two, three, four, ... regular shapes long. Complete the following tables.

Record your results in a table.

Number of pentagons	**Perimeter**
1	
2	
3	
4	
5	
6	
n	

Record your results in a table.

Number of hexagons	**Perimeter**
1	
2	
3	
4	
5	
6	
n	

Self mark exercise 1 continued on next page

Self mark exercise 1 continued

Record your results in a table.

Number of heptagons	**Perimeter**
1	
2	
3	
4	
5	
6	
n	

2 d. Try to generalize and find the perimeter of a train of ***n*** regular polygons with ***p*** - sides. Summarizing your results in a table as outlined below might be of help.

Regular shape	**Perimeter of 'train' of *n* shapes**
equilateral triangle	$n + 2$
square	$2n + 2$
pentagon	..
hexagon	..
heptagon	..
octagon	..
nonagon	..
decagon	..
p-gon	..

Check your answers at the end of this unit.

Section E: Using differences to find the next term in the sequence

The method of differences can be extended and used to predict more numbers in the sequence assuming the differences stay constant.

Terms in sequence	5		9		15		23		33		45
first differences		4		6		8		10		12	
second differences			2		2		2		2		

Working from the last row (only 2's appear there) upwards, we deduce that after the 10 in the second row we get 12, and adding 33 + 12 in the first row, gives the next term in the sequence **45**.

Which number do you expect to follow after 45?

Did you find **14** in the second row and hence 45 + 14 = 59 as the next term in the sequence.

Check that the next term in the sequence is 75.

The method of difference can help you to find more terms in a sequence even if you do not know the expression for the nth term. In more difficult situations it might only be the third, fourth or fifth differences that stay constant.

In the next activity you will find how the constant difference relates to the type of expression for the *n*th term and how you can use difference tables to predict more terms of the sequence.

Problem 2: Investigating constant differences

1. The terms of a sequence are given by $t_n = 2n - 3$

   Write down the first 6 terms, and the differences. Which differences are constant?

2. The terms of a sequence are given by $t_n = 3n^2 - 2$.

   Write down the first 6 terms, and the differences. Which differences are constant?

3. The terms of a sequence are given by $t_n = 2n^3 - 2$.

   Write down the first 6 terms, and the differences. Which differences are constant?

4. Looking at your results to question 1, 2 and 3 can you make a conjecture as to how the difference method can be used to predict whether the highest power of n in a rule is n, n^2 or n^3?

   Check your conjecture by considering the sequences with n^{th} term.

   a) $t_n = 2n^2 + 1$

   b) $t_n = n^3 - n^2$

   c) $t_n = n^2 - 2n$

   d) $t_n = n^3 - 5n$

   Write down the first 6 terms and the differences, until the differences stay constant.

5. Write down some more expression for the nth term of a sequence.

   Write down the first 6 terms and the difference, until the differences stay constant. Check the conjecture you made.

6. Now complete the following statement relating the constant second or third difference to the expression for the nth term t_n

   If the first differences in a sequence are constant then ________________

   If the 2nd differences in a sequence are constant then ________________

   If the 3rd differences in a sequence are constant then ________________

   If the p-th differences in a sequence are constant then ________________

Check your answers at the end of this unit.

Self mark exercise 2

Using differences to find the next term in the sequence.

1. Use the **difference method** to find the next three terms of these sequences

   a) 2, 5, 10, 17, 26, ... The next 3 terms are ____, ____, ____

   b) 1, 7, 17, 31, 49, ... The next 3 terms are ____, ____, ____

   c) 0, 9, 24, 45, 72, ... The next 3 terms are ____, ____, ____

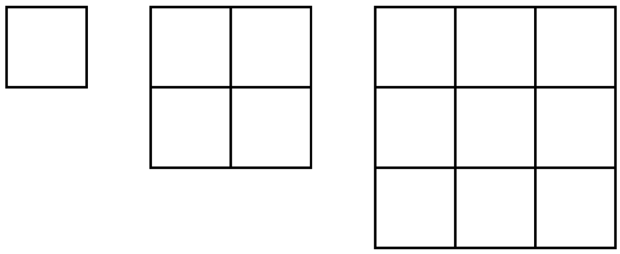
   d) 9, 15, 25, 39, 57, ... The next 3 terms are ____, ____, ____

   e) $^{-}4$, 3, 22, 59, 120, ... The next 3 terms are ____, ____, ____

   f) 0, 7, 26, 63, 124, ... The next 3 terms are ____, ____, ____

2. To make a 1×1 square using toothpicks you will need 4 toothpicks.

   For a 2 by 2 square, made of 4 smaller squares, you will need 12 toothpicks.



   a) How many toothpicks do you need to make a 3 by 3 square? A 4 by 4 square?

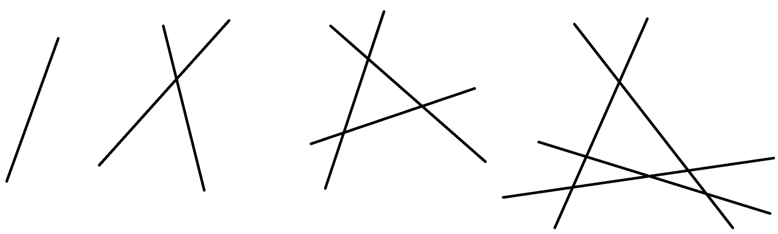
   b) Tabulate your results:

Side of the square	Number of toothpicks
1	4
2	12
3	
4	

   Use the **method of differences** to find the number of toothpicks needed for a 8 by 8 square.

   Answer: ______________ toothpicks

3. Lines are drawn to intersect all other lines in each diagram and the number of points of intersection are counted to form a sequence. Find the next four terms in the sequence using the **method of differences**.



Check your answers at the end of this unit.

Unit 2: Practice activity

1. Choose one or more of the above four activities P1, P2 or Self mark exercise 1 or 2 and work it out in more detail for use in your class as a group task. Take into account the level of understanding of your pupils (differentiate the activity if necessary for the different levels of pupils' achievement).

2. Try your activity out in the classroom with a form in which the topic is to be covered. Pupils should work together in groups of 4 to allow discussion and actively constructing and reconstructing their knowledge. As a group they should come up with ONE agreed response.

3. Write an evaluation of the lesson in which you presented the activity. Some questions you might want to answer could be: What was / were the objectives of your lesson? How did you structure the activity set to the groups and why? Was is an open or closed task? What was the criteria you used to form the groups? Did the activity work well? What was the main strength or weakness of the activity ? How was the reaction of the pupils? Did they like the activity? Did they like to work as a group? Were there any problems you have to take into account when using a similar method? What did you learn as a teacher from the lesson? Could all pupils participate? Were your objective(s) attained? Was the timing correct? What did you find out about pupils' problem solving techniques? What further activities are you planning to strengthen pupils' problem solving techniques? Were you satisfied with the outcome of the activity? Were the objectives achieved? How do you know? Was the activity different from what you usually do in the class with your pupils? How did it differ?

Present your assignment to your supervisor or study group for discussion.

Section F: Using the method of difference to find the *n*th term if the second differences are constant

Not all relationships are linear. For most of our pupils in the age range 12 - 14 linear relationships can be mastered with confidence. However a few challenging questions or problems should make them aware that the method has its limitation as several relationships are not linear. High achievers can be challenged with quadratic and even other type of relationships. In this section you will learn that the method of differences cannot only help to predict the next terms in a sequence, but also help you to find the rule for the nth term. We will be looking at sequences with second differences constant. You found in the previous activity that the expression for the *n*th term, t_n, is quadratic if second differences are constant.

Problem 3

Investigating quadratic rules for patterns.

1. A quadratic rule for a sequence is of the format $t_n = an^2 + bn + c$. There is a relationship between the second row of differences and a.

   Investigate to find this relationship.
   Start your investigation by writing down the first six terms of the following sequences and the two rows of differences.
   $t_n = 2n^2 + 3$ $\quad t_n = 3n^2 + 5n$ $\quad t_n = n^2 - 2n + 4$ $\quad t_n = 4n^2 + n + 1$

   a) Can you make a conjecture? Can you find a relationship between the differences and the coefficient a of n^2?

2. Thato investigated further to find b and c in $t_n = \boldsymbol{a}n^2 + \boldsymbol{b}n + \boldsymbol{c}$. She wrote as part of her investigation:

   $t_1 = a \times 1^2 + b \times 1 + c = a + b + c$

   $t_2 = a \times 2^2 + b \times 2 + c = 4a + 2b + c$

   $t_3 = a \times 3^2 + b \times 3 + c = 9a + 3b + c$

   $t_4 = a \times 4^2 + b \times 4 + c = 16a + 4b + c$

Terms	a + b + c		4a + 2b + c		9a + 3b + c		16a + 4b + c
1st differences		3a + b		5a +b		7a + b	
2nd differences			2a		2a		

   Thato used the difference pattern to find quadratic rules for sequences. She found a first from the 2nd differences line, then she found b from the first difference line and finally she found c using the first term t_1.

   Consider the following sequences write down the first 6 terms, first differences and second differences. Try to follow Thato's method to find the rule for the nth term in each sequence.

   (i) 3, 4, 7, 12, ...

   (ii) 0, 8, 22, 42, ...

   (iii) 5, 11, 21, 35, ...

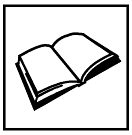
   (iv) 8, 22, 42, 68, ...

3. Can you describe how quadratic rules for patterns can be found?

   ____

   ____

Check your answers at the end of this unit.



Finding the quadratic rule for the sequence

Worked example

Find the quadratic rule for the sequence:

$^{-}4$, 3, 16, 35, ...

Terms	4		3		16		35
1st differences		7		13		19	
2nd differences			6		6		

The rule will be of the format $t_n = an^2 + bn + c$.

Then using the differences table in the previous investigation you find that:

$2a = 6$ $\quad$ $a = 3$

$3a + b = 7$ $\quad$ $9 + b = 7$ $\quad$ $b = {}^{-}2$

$a + b + c = {}^{-}4$ $\quad$ $3 + ({}^{-}2) + c = {}^{-}4$ $\quad$ $c = {}^{-}5$

The rule is $t_n = 3n^2 - 2n - 5$.

Sometimes the quadratic rule can be written down easily. Look at these examples:

1, 4, 9, 16, 25 ... $\quad$ $t_n = n^2$

2, 5, 10, 17, 26, ... $\quad$ $t_n = n^2 + 1$

2, 8, 18, 32, 50, ... $\quad$ $t_n = 2n^2$

If the rule cannot be written down easily then the above method can be used to find the quadratic rule.

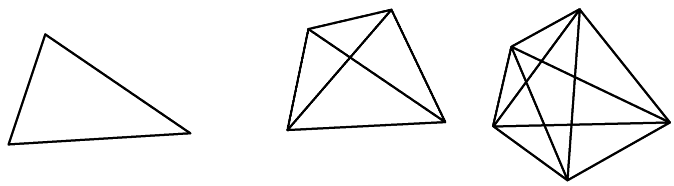
Self mark exercise 3

Finding the quadratic rule for the sequence.

1. Write down the rule for these sequences.
   a) 10, 40, 90, 160, ... t_n = ____________
   b) $^-9$, $^-6$, $^-1$, 6, ... t_n = ____________
   c) $^-2$, 1, 6, 13, ... t_n = ____________
   d) 21, 24, 29, 36, ... t_n = ____________
   e) 3, 6, 11, 18, ... t_n = ____________
   f) 5, 20, 45, 80, ... t_n = ____________

2. Find the quadratic rule for these sequences.
   a) 3, 9, 19, 33, 51, ... t_n = ____________
   b) 4, 11, 22, 37, 56, ... t_n = ____________
   c) 5, 14, 27, 44, 65, ... t_n = ____________
   d) 2, 13, 32, 59, 94, ... t_n = ____________
   e) $^-2$, 7, 22, 43, 70, ... t_n = ____________

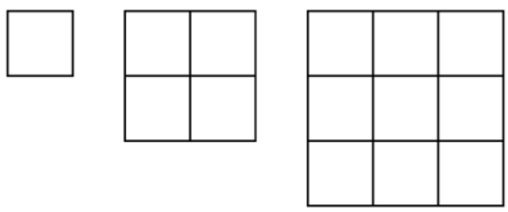
3. Diagonals are drawn from each vertex of a convex polygon to every other vertex, forming a sequence according to the number of sides of the polygon. Find the next four terms in the sequence and find an expression for the number of diagonals for an *n*-sided polygon.



Number of sides (*n*)	3	4	5	6	...
number of diagonals (*d*)	0	2	5	....	...

4. To make a 1 × 1 square using toothpicks you will need 4 toothpicks.

   For a 2 by 2 square, made of 4 smaller squares you will need 12 toothpicks.



How many do you need to make a *n* by *n* square? Answer: __________

Check your answers at the end of this unit

Section G: Games in the learning of mathematics

In this module, and others, games are used for the consolidation of concepts. In mathematics pupils are to be able to recall instantly certain facts such as the multiplication tables of 1 to 12, the first 10 powers of 2. Pupils are to apply the four basic rules on whole numbers in the range 1 - 100 , ... and also the squares of the numbers 1 - 15. In the past books used to give pages with drill and practice exercises to consolidate these, and other, concepts. Nowadays emphasis is more on processes, on relational understanding, but still consolidation is needed. For consolidation games, short challenges and puzzles form an excellent medium.

Reflection

Stop reading for a moment to think about the use of games and puzzles in the learning of mathematics.

Write down some of your thoughts.

Here are some questions that might guide your thinking.

When you were a pupil yourself at secondary school were games used in the mathematics lesson?

Can you think of some games that would be useful in the learning of mathematics?

What advantages can you see in using games as a mean to consolidate concepts?

Do you see major disadvantages? Could these be overcome? If yes, how? If no, why not?

Would you like to use (more) games in the mathematics lesson?

Now continue to read and compare with what you wrote down.

Advantages of using games

Games can:

1. Develop a positive attitude towards mathematics.
Pupils need to experience: success, excitement, satisfaction, enthusiasm, self-confidence, interest, enjoyment, active involvement. Few media are more successful than games in providing all of these experiences.

2. Consolidate mathematical concepts, facts, vocabulary, notation.
Concepts, facts, vocabulary, mathematical notation need consolidation. The traditional drill and practice episodes in a lesson are not, in general, very motivating, while a game-like environment might consolidate the concepts in a enjoyable and motivating way. In particular games can be used to consolidate mathematical facts, vocabulary, notation.

3. Develop mental arithmetic skills.
Despite the fact that calculators are a tool in the learning of mathematics, this does not dismiss the need for pupils to know basic number facts and approximate sums, differences, products and quotients. Games can address specifically these important aspects for natural numbers, integers, (decimal) fractions and percent. For example a set of dominos can be designed to consolidate the equivalence of fractions, conversion of fractions to percent or the four basic operations with integers.

4. Develop strategic thinking.
Games can encourage pupils to devise winning strategies. Can the person (i) playing first (ii) playing second always win? What is the 'best' move in a given situation? For example the game of nought and crosses. Is there a winning strategy?

5. Promote discussion between pupils and between teacher and pupil(s).
When certain games are used in the mathematics class as a learning activity, there is a need to discussion: What mathematics did you learn? Is it a good game? Can it be improved? are some of the questions to look at.

6. Encourage co-operation among pupils.
Some games requires a group playing against another group or a pair of pupils against another pair. Such games can enhance co-operation among the group or the pupils paired. They have to co-operate in order to 'win' the game.

7. Contribute to the development of communication skills.
In a game the rules need to be explained. Pupils can explain the rules to others orally, formulate rules in writing, describing strategies used to each other—activities enhancing communication.

8. Stimulate creativity and imagination.

If pupils have been playing a game for some time they can be encouraged to make a similar new game for themselves or for younger brothers and sisters. Pupils frequently devise new rules to add to or to replace the basic rules to make the game more challenging to them once the basic rules have been mastered. Pupils can also be challenged to devise variations and extensions to the game. These activities call on pupils' creativity and imagination.

9. Serve as a source for investigational work.

Games can form a source for investigational work by analysing the game and answering questions such as: Is there a best move? Can the first player always win? How many possible moves are possible? Is it a fair game? What is the maximum score I can make? What would happen if ... ? This can lead to looking at simpler cases first, tabulating results, making conjectures, testing hypotheses i.e., leading to investigational work.

Below find examples of a game, and some challenges and puzzles that can be used with pupils.

Pupils activity 1: Square routes

Game for 2 players.

Required: game board and 2 different coloured counters.

Rules: Pupils take turns in moving their counter from start to end, stepping on tiles with square numbers only. They write down the route (sequence of square numbers followed). Counters cannot be placed on the same field. The pupil reaching END first wins the game. In the next game a pupil is not allowed to take the same route as in the previous game!

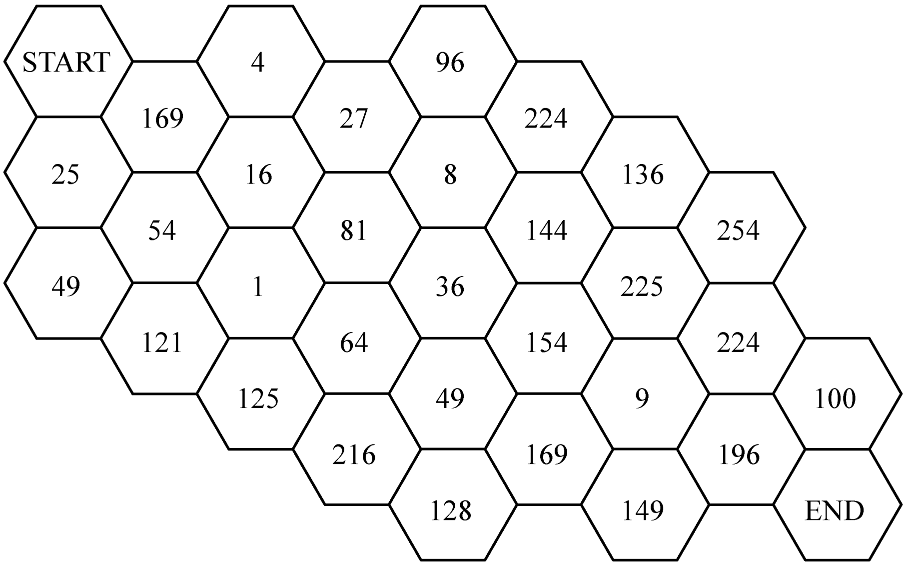
After having played the game a number of times the two players pool the routes followed and investigate how many different routes there are all together.

The pair of players finding most routes are the 'class pair winners'.

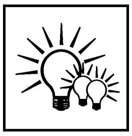
Objectives:

(i) consolidation of the squares of the whole numbers 1 – 15
(ii) develop the problem solving technique of systematic counting

How many routes can you make from start to end stepping on tiles with square numbers only?



NB: the game can be adapted for other type of numbers: prime numbers, cubes, etc.



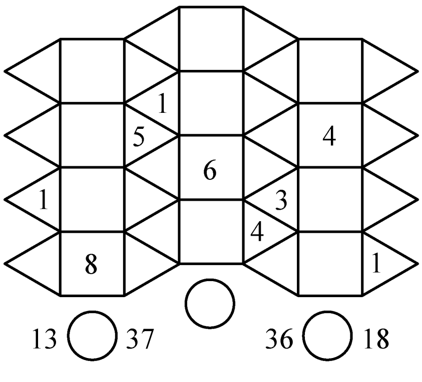
Pupils activity 2: Challenges

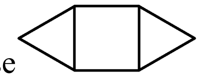
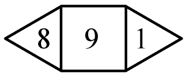
To the teacher:

Small problem solving and investigative type of questions allow teacher and pupils to assess the level the concept has been understood, as the challenges do not call on standard procedures or recall of facts only. Challenges can be used as introduction to lessons to motivate pupils or as ‘gap fillers’ - to fill some ‘left over’ minutes at the end of a lesson or topic. Challenges (pasted on a notice board) can also be set at the beginning of the week as a contest within a class or between classes. The underlying idea in all cases is to make mathematics enjoyable and interesting. This requires that challenges should be set for various levels of pupil achievement. Lower achievers need challenges just as higher achievers do.

1. Square pairs

   Objective: consolidation of the squares of the one digit numbers 1 to 9.



The diagram is made up of shapes like these  . The number in the square, when squared, gives a two digit number. These are written in the triangles. For example: with 9 in the square the shape will look like 

as the square of 9 is 81.

The column of the triangles add up to the totals shown: 13, 37, 36 and 18. The columns of the squares all add up to the same total. What is this total?

2 a. Find a triangle ABC with each angle being a square number.

b. Find a quadrilateral with all angles being a square number of degrees.

3. In $11^2 = 121$, both 11 and 121 are palindromic numbers, as they read the same from left to right as from right to left. Is the square of a palindromic number always palindromic? What about cubes, fourth powers. ... ?

Pupils activity 3: Puzzles

The distinction between a challenge and a puzzle is not very clear, but in general one could say that in puzzles a 'trial and error' is suggested while in challenges other strategies suggest themselves:

1. The square of 12 is 144. If you reverse the digits in both numbers (reading from right to left) you get another true statement: the square of 21 is 441.

   Find other two digit numbers with this property.

   Is there a three digit number with this property?

2. Find digits A, B and C such that $(BC)^2 = ABC$.
3. Find digits A, B and C such that $A^B \times C^A = ABCA$.

Unit 2: Practice activity

1. Find or design (i) a game (ii) 5 challenging questions and (iii) 5 puzzles related to the topic you are presently covering in the classroom.
2. Try out your game, challenges and puzzles in the classroom.
3. Write an evaluation of the lessons in which you presented the activities. Some questions to consider in the evaluation: What where the objectives? Did the activities succeed? If not what were the problems encountered and how do you envisage to avoid these next time? What was the reaction of the pupils? What did you learn from the activities as a teacher? Do you consider the use of games, challenges and puzzles an effective way to assist pupils in the learning of mathematics? Justify your answer.

Present your assignment to your supervisor or study group for discussion.

Section H: Problem solving method in sequences

Some mathematicians say that mathematics is about describing the patterns among numbers and shapes. It is true that in many investigations and problem solving situations data are generated, starting from simple cases, and placed in a sequence. Finding the pattern in the sequence is the next step pupils have to master.

In this unit you have learned how the method of difference may be used for finding more terms in a sequence and for finding an expression for the *n*th term in some sequences (linear and quadratic). How can some of the things you learned be presented to pupils in the classroom? Below you find three suggestions for activities you might use in the classroom. All use a problem solving, investigative approach. The cooperative model: think - pair - share could be used. In this model the pupil first thinks and goes through (part) of the question individually. Next the pupil pairs with a neighbour to compare their findings and thinking. In the next step the agreed ideas of the pair are shared with other pairs (or the whole class). Work through the following activities yourself.

Pupils activity 1: Pascal's triangle

Objective: using problem solving techniques. Start with a simpler case, tabulate results, look for a pattern and extend the pattern.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

a. Find the sum of the numbers in the 50th row

Number of row	Sum of all the numbers
1st	1
2nd	2
3rd	4
4th	.
5th	.
6th	.
50th	

b. Find the sum of all the numbers in a Pascal triangle with 50 rows.

Start with a Pascal triangle with 1 row. Next a triangle with 2 rows, etc. Tabulate results and look for a pattern.

Number of rows	Sum of all the numbers
1	1
2	3 = 1 + 2 = 4 – 1
3	7 = 1+ 2 + 4 = 8 – 1
4	.
5	.
6	.

Pupils activity 2: Paths, patterns

Objective:

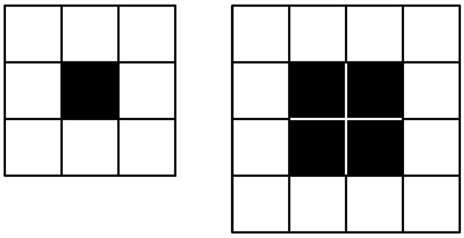
(i) Develop techniques for finding patterns in sequences and use the pattern to predict more terms in the sequence.

(ii) Develop the technique of differences for linear relationships.

Pupils work in groups in order to compare and discuss individually completed worksheets.

Three worksheets for pupils follow.

1. The diagram illustrates a path surrounding a flower bed paved with square slabs.



a. If the flower bed is 1 m by 1 m the number of square slabs needed is ___.

b. If the flower bed is 2 m by 2 m the number of square slabs needed is ___.

c. On squared paper make a scale drawing of a path around a 3 m by 3 m flower bed.

d. How many square slabs are needed for a 3 m by 3 m flower bed?

Needed are _______ square slabs.

e. What happens if the flower bed increases in size to 4 m by 4 m, 5 m by 5 m, etc. How many square slabs will be needed in each case?

Record your results in a table.

Size flower bed	Number of square slabs
1×1	8
2×2	
3×3	
4×4	
5×5	
6×6	

f. What pattern do you notice in the last column?

g. Look at the **differences** between consecutive numbers of square slabs and see if you will be able to extend your table.

How many slabs are needed for a 7 m by 7 m flower bed? ______

A 10 m by 10 m flower bed? _______

A 50 m by 50 m flower bed? ________

h. How many slabs are needed for an n by n flower bed? _________

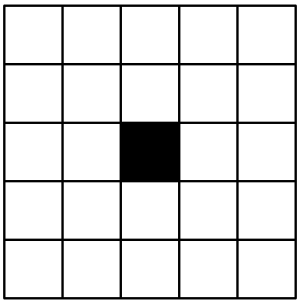
Compare your answers with the students in your group. Discuss differences and agree on a rule to find the number of square slabs needed for any square flower bed.

Your rule:

The number of slabs needed for an n by n flower bed is _______________.

2a. What happens if the path is **2 slabs** wide?

Draw, on square cm paper, the first three paths. The first one is illustrated in the diagram.



Record your results in a table.

Size flower bed	Number of square slabs
1×1	24
2×2	
3×3	
4×4	
5×5	
6×6	

b. Look at the **differences** between consecutive numbers of square slabs and see if you will be able to extend your table.

How many slabs are needed for a 7 m by 7 m flower bed? ______

c. A 10 m by 10 m flower bed? _______

d. A 50 m by 50 m flower bed? ________

e. An n by n flower bed?

Compare your answers with the students in your group. Discuss differences and agree on a rule to find the number of square slabs needed for any square flower bed.

Your rule:

The number of slabs needed for an n by n flower bed is ________________ .

3a. What number of slabs will be needed if the path is 3 slabs wide?

Draw some of the paths surrounding the flower bed and tabulate your results.

Record your results in a table.

Size flower bed	Number of square slabs
1×1	48
2×2	
3×3	
4×4	
5×5	
6×6	

b. Look at the **differences** between consecutive numbers of square slabs and see if you will be able to extend your table.

How many slabs are needed for a 7m by 7m flower bed? ______

c. A 10 m by 10 m flower bed? _______

d. A 50 m by 50 m flower bed? ________

e. An n by n meter flower bed?

> **Compare your answers with the students in your group. Discuss differences and agree on a rule to find the number of square slabs needed for any square flower bed.**

Your rule:

The number of slabs needed for an n by n flower bed is _______________ .

4. Did you find the following results in your tables?

8, 12, 16, 20, 24, 28, 32, 36, ... $4n + 4 = 4(n + 1)$

24, 32, 40, 48, 56, 64, 72, ... $8n + 16 = 8(n + 2)$

48, 60, 72, 84, 96, 108, ... $12n + 36 = 12(n + 3)$

Look at the number of square slabs needed for an n by n flower bed with width 1, 2 and 3 slabs.

a. What do you think will be the number of slabs needed surrounding an n by n flower bed if the path is 4 slabs wide ?

The number of square slabs needed is ____________.

b. What if the path is 5 slabs wide?

The number of square slabs needed is ____________.

c. What if the path is 6 slabs wide?

The number of square slabs needed is ____________.

d. What if the path is s slabs wide?

The number of square slabs needed is ____________.

Introduction.

Perhaps you had some difficulties in finding the expressions for the number of slabs surrounding a n metre by n metre flower bed.

You should have found: $4n + 4 = 4(n + 1)$, for a path of width 1 slab
$8n + 16 = 8(n + 2)$, for a path of width 2 slabs
$12n + 36 = 12(n + 3)$, for a path of width 3 slabs
$16n + 64 = 16(n + 4)$, for a path of width 4 slabs
$20n + 100 = 20(n + 5)$, for a path of width 5 slabs
...
...
$4sn + 4s^2 = 4s(n + s)$, for a path of width s slabs.

In a table.

Size of paths	Number of square slabs needed to surround an *n* by *n* flower bed
1 slab wide	$4(n + 1)$
2 slabs wide	$8(n + 2)$
3 slabs wide	$12(n + 3)$
4 slabs wide	$16(n + 4)$
5 slabs wide	$20(n + 5)$
...	...
s slabs wide	$4s(n + s)$

Let's look how these rules can be found.

You found the following sequences:

8, 12, 16, 20, 24, 28, 32, 36, ... $4n + 4 = 4(n + 1)$

24, 32, 40, 48, 56, 64, 72, ... $8n + 16 = 8(n + 2)$

48, 60, 72, 84, 96, 108, ... $12n + 36 = 12(n + 3)$

1a. What is the difference between terms in the first sequence?

Answer ______

b. Where do you find this constant difference back in the rule for the number of slabs needed to surround an n metre by n metre flower bed?

____

c. What is the difference between terms in the second sequence?

Answer _______

d. Where do you find this constant difference back in the rule for the number of slabs needed to surround an n metre by n metre flower bed?

e. What is the difference between terms in the third sequence?

Answer ______

f. Where do you find this constant difference back in the rule for the number of slabs needed to surround an n meter by n meter flower bed?

____

The constant differences seem to be important as we find them back in our rule as the coefficient of *n*.

How to find the rule.

Consider the following sequence:

	7		10		13		16		....
(First) differences		3		3		3		3	

We expect now the rule to be of the form:

3*n* + (number)

As the first number is 7, taking for $n = 1$ $3 \times 1 + \text{(number)} = 7$

Solving for (number): (number) = 7 – 3 = 4

Our rule for the n^{th} term in the sequence is $t_n = 3n + 4$

Check that this gives the correct terms.

As our rule is of the form $y = ax + b$ (when y is plotted against x it gives a LINE graph) we say that $t_n = 3n + 4$ is a **linear** rule.

Worksheet 3 Using differences to find linear rules of sequences

1. Find rules that generate the terms of the following sequences. Write the rules as t_n = ________

Test the rule by finding the next few terms.

a. 3, 6, 9 ,12, ... t_n = ___________

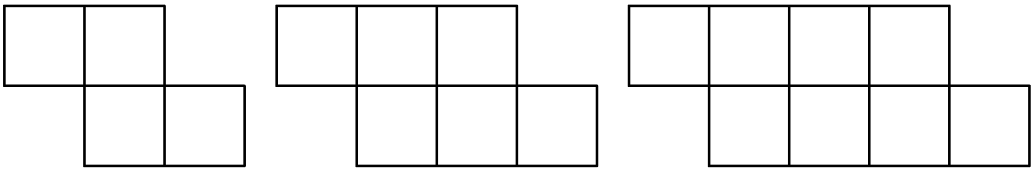
b. 4, 9, 14, 19, ... t_n = ___________

c. 8, 14, 20, 26, ... t_n = ___________

d. $^{-}6$, $^{-}2$, 2, 6, ... t_n = ___________

e. $^{-}4$, $^{-}6$, $^{-}8$, $^{-}10$ ... t_n = ___________

2. The following illustrates a growing pattern of unit squares. In the first pattern there are 4 unit squares, in the second 6, etc.



1 2 3

a. Complete this table:

pattern number	1	2	3	4	5	6
number of squares	4	6	.	.	.	.

b. How many unit squares are there in the 10th diagram if the pattern continues?

c. How many unit squares are there in the 100th diagram if the pattern continues?

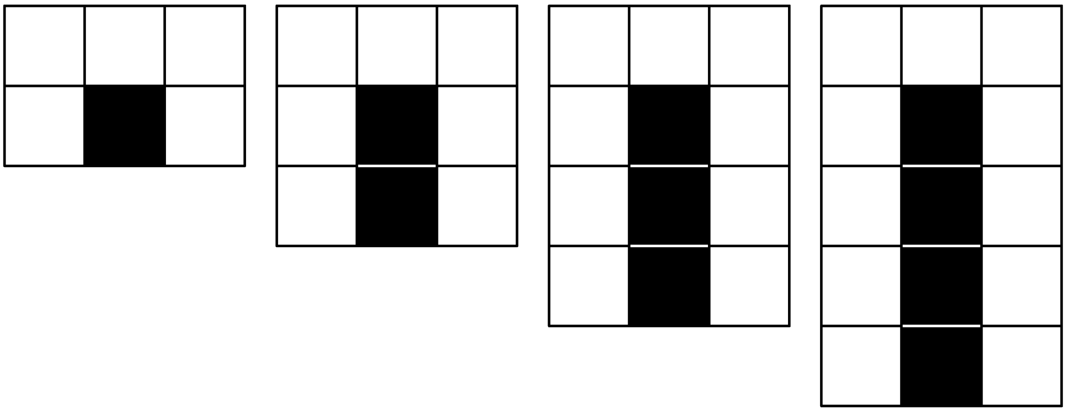
d. How many unit squares are there in the 500th diagram if the pattern continues?

e. How many unit squares *s* are there in the *n*th diagram if the pattern continues?
s = ___________

f. Investigate the perimeter of each pattern and find an expression for the perimeter (P) of the *n*th diagram.

3. A path is made using white and black tiles. The following diagram illustrates paths of length 2, 3, 4 and 5 metres long.

a. Draw the next two patterns on **square cm paper**.



b. Complete this table:

Path length	No. of black squares	No. of white squares
2	1	5
3	2	
4		
5		
6		
7		
8		
9		
p		

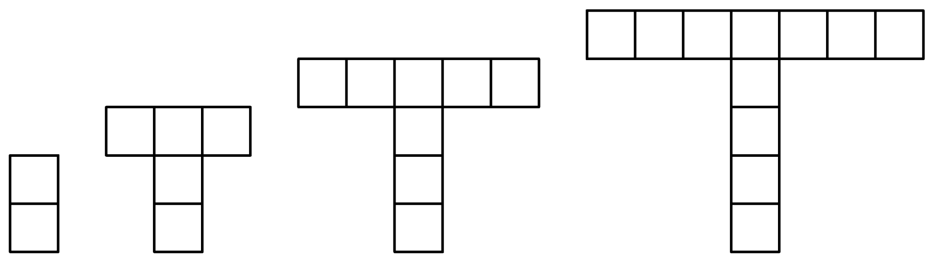
c. Find a formula relating the number of white tiles (W) and the number of black tiles (B).
W = _______________

d. If there are 72 black tiles how many white tiles are needed?

e. If there are 209 white tiles in a path how many black tiles are there?

4. A 'growing' shape is made using tiles. The following diagram illustrates the first four shapes.

a. Draw the next two patterns on **square cm paper**.



b. Complete this table:

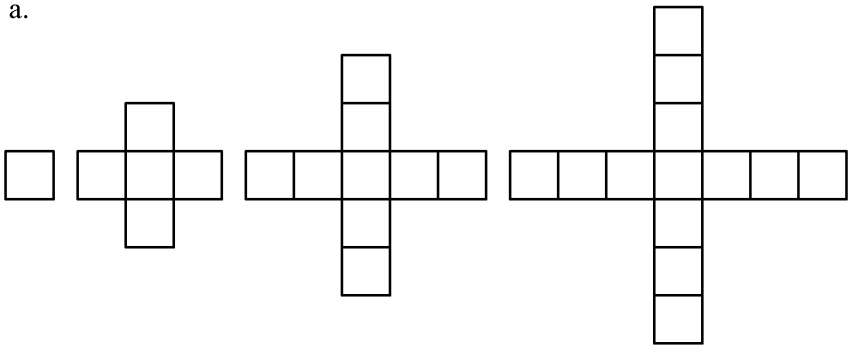
Pattern number	**No. of square tiles used**
1	2
2	5
3	
4	
5	
6	
7	
8	
n	

c. Find a formula relating the number of tiles (t_n) needed to make the nth pattern.
t_n = _______________

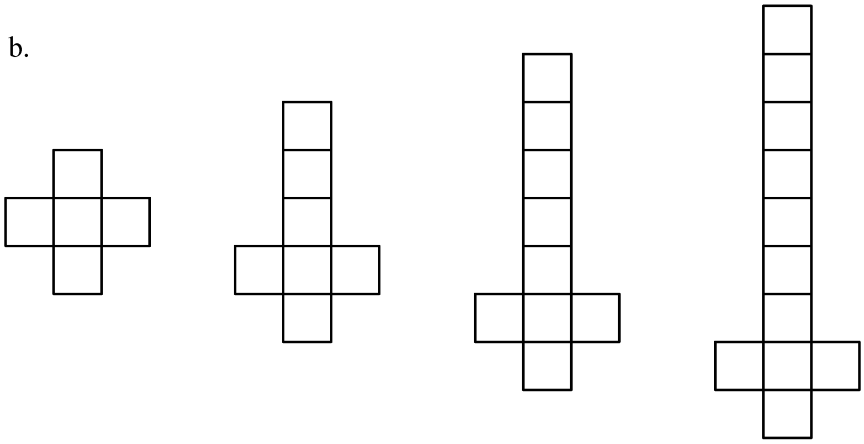
d. If there are 78 tiles what pattern number can be made?

5. Repeat question 4 a, b, and c for the next 'growing' tiles patterns.

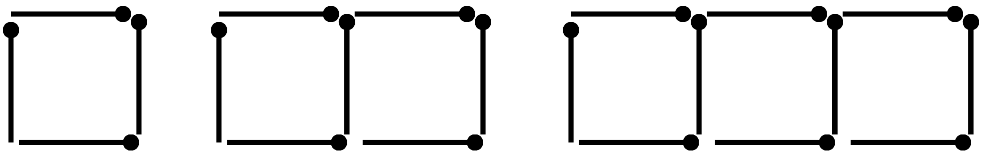
a.



b.



6. Match sticks are used to make patterns. In the following patterns find:
   a. Number of match sticks in the perimeter of the *n*th pattern.
   b. How many match sticks are needed to make the *n*th pattern?



Record your results in a table.

Pattern number	1	2	3			*n*
No. of match sticks in the perimeter	4					
No. of matches to make the pattern		7				

Pupils activity 3: Power sequences - The Tower of Hanoi & spread of rumours

To the teacher:

On the pupils' worksheet which follows, the problem is stated without suggesting the steps. For weaker pupils you might have to adapt the worksheet.

Objective: To develop and use the 4 Polya steps (1. understand the question 2. make a plan 3. carry out the plan 4. evaluate the solution) in problem solving (Polya, 1957).

Notes to the teacher on Tower of Hanoi

Pose the problem first for a 2 disc situation. Be sure pupils understand how the discs are to be moved. A model is needed. Use 2 paper circles of different sizes; instead of the three needles use three pieces of paper to mark the position of the first, second and third needle. Coins or jar lids could also be used for the discs. Let pupils work on the 2 disc model and extend to a 3, 4 and 5 disc model. Some might find the 31 moves needed for the 5 discs case.

Now tell the story of the tower. Ask pupils to guess and give an explanation of their guess. Write down the guesses on the board. Suggest that they record the results obtained so far and start looking for a pattern.

Tower of Hanoi

No. of discs	No. of moves
1	1
2	3
3	7
4	15
5	31
.	.
.	.
n	$2^n - 1$

Pupils should recognize the same pattern as in the Pascal triangle and rumour problem. It is the same pattern as the sum of the powers of 2. This needs further exploring: how many times is each disc moved? Number the discs and let pupils record the number of moves of each disc. Pupils are to be encouraged to keep record of the moves of each disc. (The bottom disc will make only one move, the disc above the bottom one will make 2 moves etc. leading to the summation of the powers of two.)

The number of moves for the 64 golden discs in the temple complex is $2^{64} - 1 = 18\ 446\ 744\ 073\ 709\ 551\ 615$ moves and takes the same amount of minutes; converted to years this gives about 3.5×10^{15} years, a long time before the end of the world!

Notes to the teacher about the spread of rumours

Pose the problem. Be sure pupils understand how the rumour is spread. You may want to act it out in the classroom.

Ask pupils to guess and for an explanation of their guess. Write down the guesses on the board. Having pupils guess before formally pursuing the problem gets them interested in the problem: they have interest in the outcome! (Did I guess right?)

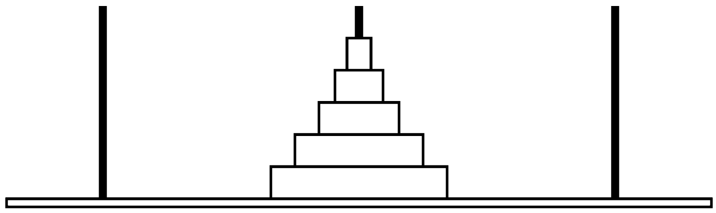
Allow pupils to explore the problem in groups and ensure that the groups understand the problem. Suggest that they keep record of their findings in an organized way. If a group finishes early set some extension questions. (On what day will more than 10 000 be told the rumour? On what day will more than 100 000 have heard the rumour?)

After the groups have finished, call on different groups to discuss their solution. A group will surely have used a pattern approach. Set up a table on the chalk board. As you generate the table, ask pupils to predict the next entry. Stop the pattern on day 12. Encourage pupils to look for a pattern by examining the second column as the powers of 2. Pupils should also express the generalization in words.

Day	No. of new people hearing the rumour	Total number of people who heard the rumour
1	$2 = 2^1$	$3 = 4 – 1 = 2^2 – 1$
2	$4 = 2^2$	$7= 8 – 1= 2^3 – 1$
3	$8 = 2^3$	$15 = 16 – 1= 2^4 – 1$
4	$16 = 2^4$	$31= 32 – 1= 2^5 – 1$
5	32 = ...	63 = ...
6	64 = ...	127 = ...
.	.	.
10	1024 = ...	2047 = ...
11	2048 =....	4095 = ...
12	4096 =....	8191= ....
.		.
.		.
.	.	.
n	2^n	$2^{n+1} – 1$

The Tower of Hanoi

The Tower of Hanoi is in the great temple of Benares in India. The story goes that there are three diamond needles and sixty-four golden discs, graduated in size. The largest disc is at the bottom, the smallest at the top on one of the needles. (See the illustration for a tower with only five discs.)



The monks from the temple are to move the discs one at the time to another needle. A **larger disc can never be placed on a smaller disc**. What is the fewest number of moves necessary to move the entire stack of discs to another needle, so that they are again arranged from largest at the bottom to smallest at the top?

When all the discs have been transferred the world will come to an end. If the monks started at the beginning of mankind, how close are we to having the world end? (Assume, as the discs are very large and heavy, it takes one minute per move).

Rumours

On the 5th of October a rumour is started that pupils in JSSs will have to pay school fees as from the next school year. On the first day—October 5th—one pupil tells the rumour to two other pupils with the instruction that each is to spread the rumour to two more pupils the next day and that each of these pupils is to repeat the rumour to two more pupils on the third day, and so on. So on the first day three pupils heard the rumour, on the second day four more will have heard, on the third day eight more will have heard and so on.

How many new pupils will be hear the rumour on the tenth day? The 50th day? The *n*th day?

How long will it take before all pupils in your school have heard the rumour ?

Self mark exercise 4

1. A square paper is folded repeatedly. Investigate (i) the number of regions formed and (ii) the area of the smallest region.

Check your answers at the end of this unit.

Unit 2, Assignment 3

1. Choose one of the pupils' three activities suggested in section G. Work the activity out in more detail so you can use it in your class with your pupils. Take into account the level of understanding of your pupils (differentiate the activity if necessary for the different levels of pupils' achievement).

2. Try out your activity in the classroom.

3. Write an evaluation of the lesson in which you presented the activity. Some questions you might want to answer could be: What were the strengths and weaknesses? What needs improvement? How was the reaction of the pupils? What did you learn as a teacher from the lesson? Could all pupils participate? Were your objective(s) attained? Was the timing correct? What did you find out about pupils' investigative abilities? What further activities are you planning to strengthen pupils' problem solving and investigative work? Were you satisfied with the outcome of the activity?

Present your assignment to your supervisor or study group for discussion.

Summary

This unit has covered the traditional subject of sequences. It has also introduced the much more recent teaching concepts of:

- teaching maths through games
- having pupils work on problems jointly, then present how they did it to the class ("Think Pair Share")

When guided well by the teacher, approaches like those can have remarkable benefits to the quality of learning in a mathematics classroom.

Unit 2: Answers to self mark exercises

Problem 1

1a. 4
b. Rule: multiply the number of the term by 4 and subtract 1
c. coefficient of n

2a. 7
b. Rule: multiply the number of the term by seven and add 1
c. coefficient of n

3a. $^{-}5$
b. Rule: multiply the number of the term by $^{-}5$ and add 35
c. coefficient of n.

Self mark exercise 1

1a. $t_n = 4n + 2$ b. $t_n = 5n - 3$ c. $t_n = {}^{-}3n + 55$ d. $t_n = 4n - 9$
e. $t_n = {}^{-}2n$ f. $t_n = 0.5n + 2$ g. $t_n = \frac{-3}{4}n + 6\frac{3}{4}$

2a. $t_n = n + 2$ b. $t_n = 2n + 2$ c. $t_n = 3n + 2$, $t_n = 4n + 2$, $t_n = 5n + 2$
d. $t_n = (p - 2)n + 2$

Problem 2

1. $^{-}1$, 1, 3, 5, 7, 9 Constant 1st differences (of 2)

2.

1	10	25	46	73	106	...
9	15	21	27	33	...	
6	6	6	6			

2nd differences are constant (6)

3.

0	14	52	126	248	430
14	38	74	122	182	
24	36	48	60		
12	12	12			

3rd differences are constant (12).

4/5/6. If the first difference is constant the rule starts with an.
$t_n = an + b$ $(a \neq 0)$.

If the second difference is constant the rule starts with an^2.
$t_n = an^2 + bn + c$ $(a \neq 0)$

If the third difference is constant the rule starts with an^3.
$t_n = an^3 + bn^2 + cn + d$ $(a \neq 0)$

If the pth difference is constant the rule will start with an^p $(a \neq 0)$.

Self mark exercise 2

1a. 37, 50, 65 b. 71, 97, 127 c. 105, 144, 189 d. 79, 105, 135

e. 211, 338, 507 f. 215, 342, 511

2a/b. 24, 40 For 8 × 8 needed 144

3. 0, 1, 3, 6, 10, 15, 21, 28, 36

Problem 3

1a. Conjecture (i) the second difference is constant (ii) the coefficient of n^2 is half the second difference.

2. (i) $n^2 - 2n + 4$ (ii) $3n^2 - n - 2$ (iii) $2n^2 + 3$ (iv) $3n^2 + 5n$

3. The general term is of the form $t_n = an^2 + bn + c$.

(i) Make a difference table

(ii) a is half the second difference

(iii) Use the first difference row to find b

(iv) Use the first term to find c

Self mark exercise 3

1a. $t_n = 10n^2$ b. $t_n = n^2 - 10$ c. $t_n = n^2 - 3$ d. $t_n = n^2 + 20$
e. $t_n = n^2 + 2$ f. $t_n = 5n^2$

2a. $t_n = 2n^2 + 1$ b. $t_n = 2n^2 + n + 1$ c. $t_n = 2n^2 + 3n$ d. $t_n = 4n^2 - n - 1$
e. $t_n = 3n^2 - 5$

3. $t_n = \frac{1}{2}n(n-3)$

4. $t_n = 2n(n+1)$

Self mark exercise 4

Number of folds	Number of regions	Area of smallest region
0	1	1
1	2	$\frac{1}{2}$
2	4	$\frac{1}{4}$
..	..	..
n	2^n	$\frac{1}{2^n}$

Unit 3: Polygonal numbers

Introduction to Unit 3

The number of dots required to make 'growing' dot patterns of (regular) polygons forms a sequence. The numbers in the sequence and the numbers in different sequences have multiple relationships which can be illustrated geometrically and proved algebraically. Figurative numbers therefore form a rich topic to explore both algebra and geometry.

Purpose of Unit 3

The aim of this unit is to:

- apply the technique to analyse sequences developed in Unit 2 to figurative numbers
- link geometric and algebraic representation of numbers
- offer a motivating context for working with algebra

Objectives

When you have completed this unit you should be able to:

- list and illustrate the first five figurative numbers (triangular numbers, square numbers, pentagonal numbers, ..., *p*-gonal numbers)
- express in algebraic form the *n*th polygonal number using the method of differences
- illustate relationships among polygonal using geometrical dot patterns
- prove algebraically relationships among polygonal numbers
- set activities for pupils to learn about figurative numbers and their relationships

Time

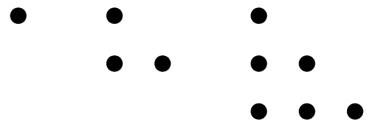
To study this unit will take you about four hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topic.

Unit 3: Polygonal numbers

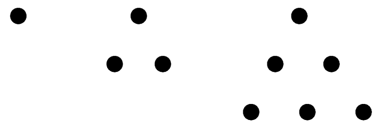
Section A: Triangular, square, pentagonal, hexagonal numbers

If we make dot patterns of ‘growing’ patterns of regular polygons, the number of dots needed form sequences. These sequence of numbers are called triangular numbers, square numbers, pentagonal numbers, hexagonal numbers, etc. In general we could speak about *p*-gonal numbers, where *p* represent the number of sides of the regular polygon (the polygons don’t really have to be regular, but the patterns look more attractive when placed in regular polygonal arrays).

Triangular numbers can be illustrated with a dot pattern giving a triangle. Below are two formats. The first places the dots to form right-angled isosceles triangles, the second pattern places the dots to form equiangular triangles.



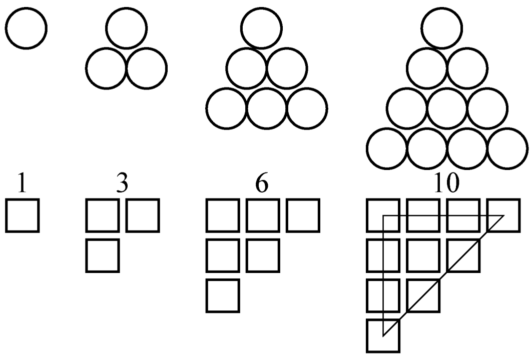
1 3 6



1 3 6

The triangular numbers are 1, 3, 6, 10, 15, 21, ...

Circles or squares can be used as well in illustrating the triangular numbers. See the following diagram. Joining centers of ‘corner’ circles or squares suggest the ‘triangular’ form.



Self mark exercise 1

1. The triangular numbers are 1, 3, 6, 10, 15, 21, ... List the next 3 triangular numbers.
2. Continue the dot patterns of the previous page by drawing the dot patterns to represent the next 3 triangular numbers in each of the sequences.
3. Express in words how the terms in the sequence are obtained.
4. $t_1 = 1$, $t_2 = 3$, $t_3 = 6$, ... Can you find an expression for t_n, the nth triangular number in the sequence?

Check your answers at the end of this unit.

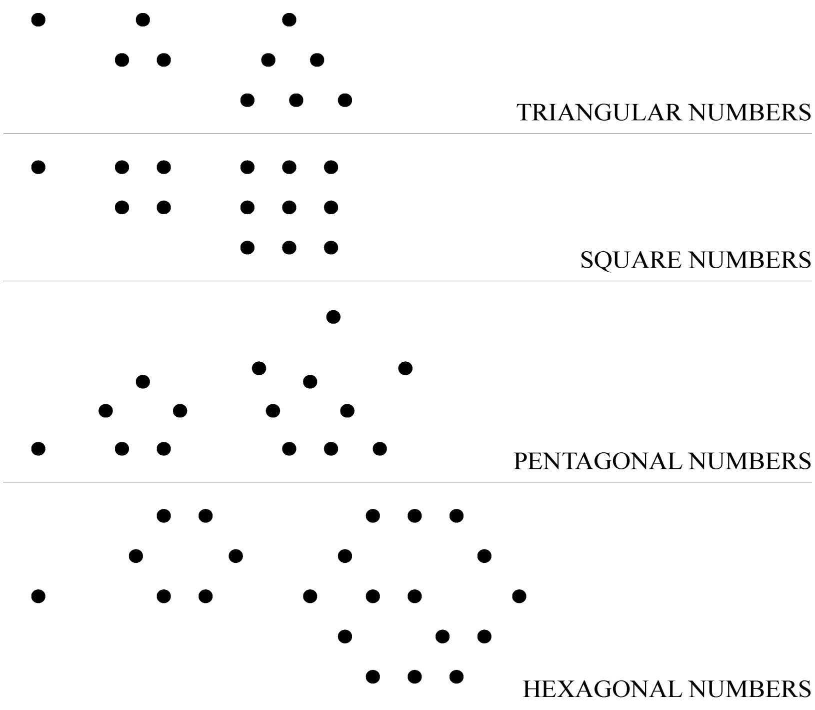
Did you use the method of differences or did you use another method?

Did you see that

$$t_1 = 1 = \frac{1\times 2}{2},\quad t_2 = 3 = \frac{2\times 3}{2},\quad t_3 = 6 = \frac{3\times 4}{2},\quad t_4 = 10 = \frac{4\times 5}{2},\quad ....$$

continuing the pattern gives as for $t_n = \frac{n(n+1)}{2} = \frac{1}{2}n(n+1)$.

Here is a sequence of figurative number patterns.



Looking at the pentagons you can see that the second pentagonal numbers (5) are represented by the vertices of a pentagon with side one unit. The next pentagonal number (12) is the previous pentagon and an enlargement of it by factor 2 (the bigger pentagon with side of two units). In the next step the original one unit sided pentagon is enlarged to a pentagon with side 3 units. Hexagonal number dot patterns are made in a similar way.

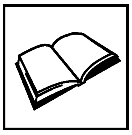
Self mark exercise 2

1. Find expressions for the *n*th triangular, square, pentagonal number and hexagonal number, drawing more dot patterns if needed to find more terms of the sequence.
2. Express in words the rule to find the next triangular, square, pentagonal and hexagonal number in a sequence once you found the first 5 or 6 terms.
3. Make dot patterns in the form of 'growing' heptagon and octagons to represent heptagonal and octagonal numbers.
4. Using the method of differences find at least the first 6 terms in each sequence of heptagonal and octagonal numbers.
5. Find an expression for the *n*th term in each of these sequences.
6. Place all your results in a table. Look for patterns in the rows and in the columns.

Type of number	Sequence of numbers	Expressions for n^{th} term
triangular	1, 3, 6, 10, 15, ...	
square		
pentagonal		
hexagonal		
heptagonal		
octagonal		

Check your answers at the end of this unit.

The table you were to complete on the previous page should look as below.



Type of number	Number of sides	First terms in the sequence	nth term: t_n
triangular	p = 3	1, 3, 6, 10, 15, 21, 28, ...	$\frac{1}{2}n(n+1)$
square	p = 4	1, 4, 9, 16, 25, 36, 49, ...	$n^2 = \frac{1}{2}n(2n+0)$
pentagonal	p = 5	1, 5, 12, 22, 35, 51, 70,...	$\frac{1}{2}n(3n-1)$
hexagonal	p = 6	1, 6, 15, 28, 45, 66, 91, ...	$\frac{1}{2}n(4n-2) = n\,(2n-1)$
heptagonal	p = 7	1, 7, 18, 34, 55, 81, 112, ...	$\frac{1}{2}n(5n-3)$
octagonal	p = 8	1, 8, 21, 40, 65, 96, 133, ...	$\frac{1}{2}n(6n-4) = n(3n-2)$
nonagonal	p = 9		
decagonal	p = 10		
p-gonal	p		

You are now to study the table in more detail to find the many patterns and relationships in order to complete the next and last row: the data for the nonagonal, decagonal and p-gonal numbers, the number of dots that can be placed such that they form polygons with p sides. The following questions should guide you.

Self mark exercise 3

1 a. What is the first term in each sequence?

b. What will be the first nonagonal, decagonal, p-gonal number? Write them in the table.

2. Now look at the second term of each sequence: 3, 4, 5, 6, ... and compare also with the number of sides of the polygon in each case. Assuming the pattern continues you can write down the second nonagonal, decagonal number and p-gonal number.

3. Now move to the third term in each sequence. The third terms form the sequence 6, 9, 12, 15, 18, 21 ... This allows you to find the third nonagonal and decagonal number.

4. Lets look for the third p-gonal number. Here you must be careful because in the sequence 6, 9, 12, 15, ... the general term would be $3n + 3$ (Check!). But the first term 6 goes together with $p = 3$, the second term 9 goes with $p = 4$. Tabulated we have:

Number of sides p	3rd p-gonal number
3	$6 = 2 \times 3$
4	$9 = 3 \times 3$
5	$12 = 4 \times 3$
6	$15 = 5 \times 3$
7	$18 = 6 \times 3$
8	...
9	....
10	.....
p	$(........) \times 3$

Complete the table.

5. Following the method as used in **3** and **4** find the first seven nonagonal, decagonal and p-gonal numbers.

Self mark exercise 3 continued on next page

Self mark exercise 3 continued from previous page

6. Finally look at the last column, the nth term in each sequence. Tabulate the values of p and the nth terms

Number of sides p	nth p-gonal number
3	$\frac{1}{2}n(n+1)$
4	$\frac{1}{2}n(2n+0)$
5	$\frac{1}{2}n(3n-1)$
6	$\frac{1}{2}n(4n-2)$
7	$\frac{1}{2}n(5n-3)$
8	$\frac{1}{2}n(6n-4)$
9	....
10	....
p	....

All nth terms start with ......

Now look at the expression in the brackets. The coefficient of n is ... less than the value of p.

Finally the last number in the brackets 1, 0, $^-1$, $^-2$, ... forms a sequence with a constant difference of $^-1$ so the general term starts with ^-p. The format is ^-p + (...). To find what is on the dots in the brackets remember that for $p = 3$ you must get 1.

Now you can find what is to be put on the place of the dots in the brackets: ....

Using all this information you can now complete the table for the nth term of the nonagonal numbers and decagonal numbers and also for the nth term in the sequence of p-gonal numbers.

Check your answers at the end of this unit.

Section B: Relationships among polygonal numbers

The relationships among the polygonal numbers are numerous. Apart from deriving relationships using algebra (deductive knowledge, illustrated below under **III**) relationships can also be illustrated (inductive knowledge) using dot, circular or square patterns. This is done below in part **I** and **II**.

Let's look at a few relationships first before you try some on your own.

Statement:
The sum of two consecutive triangular numbers is a square number.

I. Inductive method using patterns
The triangular numbers are 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ...

Tabulate sum of 2 consecutive triangular numbers:

$1 + 3 = 4 = 2^2$

$3 + 6 = 9 = 3^2$

$6 + 10 = 16 = 4^2$

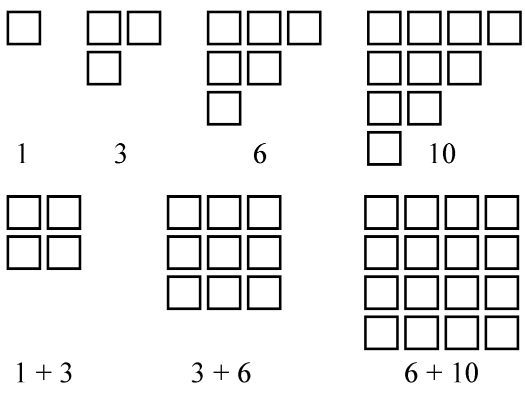
$10 + 15 = 25 = 5^2$

...
...
...

Write down some more lines in this pattern.

The conjecture is: the sum of two consecutive triangular numbers is a square number.

II. Inductive method using representations of the triangular numbers by patterns of unit squares.



Triangular numbers

Sum of two consecutive triangular numbers

The diagrams above illustrate that two consecutive patterns representing two consecutive triangular numbers can be arranged to form a square pattern. The conjecture is therefore: the sum of two consecutive triangular numbers is a square number.

III. Algebraic proof

The nth triangular number is $t_n = \frac{1}{2}n(n + 1)$

The next triangular number $t_n + 1$ is obtained by replacing our n in the previous line by $(n + 1)$.

Hence $t_{n+1} = \frac{1}{2}(n+1)[(n + 1) + 1] = \frac{1}{2}(n+1)(n+2)$

The sum of the two consecutive triangular numbers is

$t_n + t_{n+1} = \frac{1}{2}n(n+1) + \frac{1}{2}(n+1)(n+2)$	common factor $(n + 1)$ outside the brackets
$= \frac{1}{2}(n + 1)[n + n + 2]$	simplifying
$= \frac{1}{2}(n+1)(2n+2)$	common factor 2 outside brackets
$= \frac{1}{2}(n+1) \times 2 \times (n + 1)$	simplifying
$= (n + 1)^2$	a square number!

Statement: Every hexagonal number is a triangular number.

I. Inductive verification.

The hexagonal numbers are 1, 6, 15, 28, 45, ... $h_n = \frac{1}{2}n(4n - 2)$

Comparing with the triangular numbers

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ... $t_n = \frac{1}{2}n(n + 1)$

you can note that the first hexagonal number = first triangular number

$h_1 = t_1$

the second hexagonal number = third triangular number

$h_2 = t_3$

the third hexagonal number = fifth triangular number

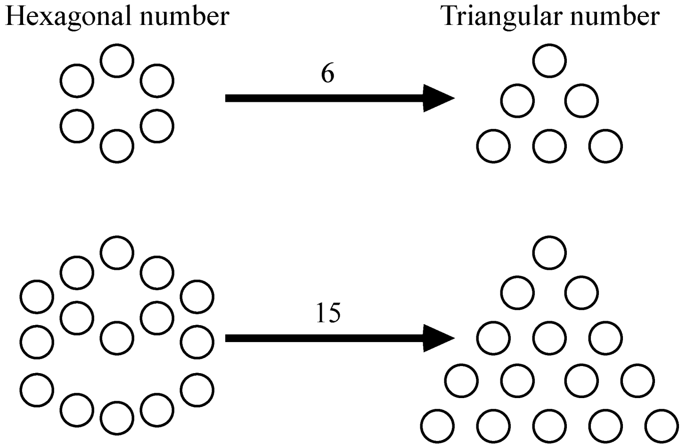
$h_3 = t_5$

$h_4 = t_7$

$h_5 = t_9$

Or generalizing $h_n = t_{2n-1}$. The nth hexagonal number is equal to the $(2n - 1)$th triangular number. So each hexagonal number is a triangular number.

II. Using patterns to illustrate that each hexagonal number equals to a triangular number. ("Pushing in the circles from the bottom of the hexagon")



III. Algebraic proof

The nth hexagonal number is $h_n = \frac{1}{2}n(4n-2)$ (factorise)

$= \frac{1}{2}n \times 2 \times (2n-1)$ (rearrange)

$= \frac{1}{2}(2n-1)(2n)$ (writing as triangular number)

$= \frac{1}{2}(2n-1)[(2n-1)+1]$

This is the expression for the $(2n-1)$th triangular number.

Statement:
Every pentagonal number is one-third of a triangular number.

I. The statement is equivalent to saying that 3 times a pentagonal number is a triangular number.

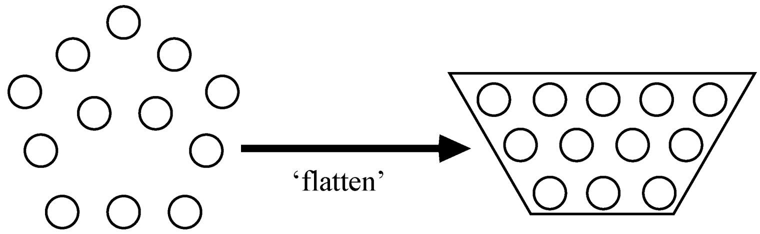
The pentagonal numbers are 1, 5, 12, 22, 35, ...

Three times these numbers give the sequence 3, 15, 36, 66, 105, ... these are respectively the triangular numbers t_2, t_5, t_8, t_{11}, ...

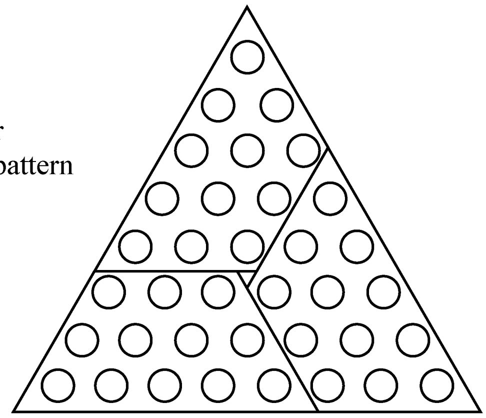
The pattern is therefore
$3 \times p_1 = t_2$
$3 \times p_2 = t_5$
$3 \times p_3 = t_8$
$3 \times p_4 = t_{11}$
Or generalizing $3 \times p_n = t_{3n-1}$

II. Geometrically flatten the ‘roof’ of each pentagonal number to make rows of dots / circles to make an equilateral trapezoid (bucket shape). Three of these fit together to make a triangle.

This is illustrated for the pentagonal number 12.



Fitting three together
to form a triangular pattern



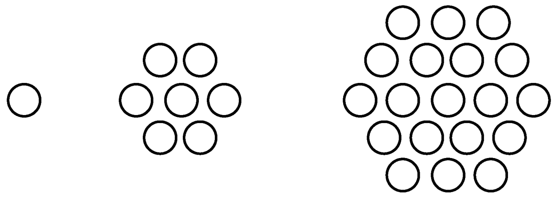
III. Algebraic proof

$$3 \times \frac{1}{2}n(3n-1) = \frac{1}{2}(3n-1)(3n) = \frac{1}{2}(3n-1)[(3n-1)+1]$$

3 times the *n*th pentagonal number gives the $(3n-1)$th triangular number.

Self mark exercise 4

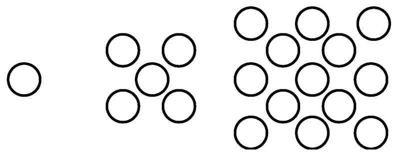
1. Illustrate (inductively) and prove (deductively) that doubling any triangular number (except the first) gives a rectangular number.
2. Find the first four dodecagonal (12 sided polygonal) numbers.
3. A p-gonal number is 40. Find p.
4. Illustrate and prove that pentagonal numbers (except the first) are the sum of a triangular number and a square number.
5. The diagram illustrates the first three **Hex numbers**—centred hexagonal numbers.



   a. Find and illustrate the next two hex numbers.

   b. Use the method of differences to find an expression for the nth hex number.

6. The diagram illustrates the first three **central square numbers**.



   a. Find and illustrate the next two central square numbers.

   b. Use the method of differences to find an expression for the nth central square number.

7. Prove and illustrate that each central square number is the sum of two consecutive square numbers.

Check your answers at the end of this unit.

Section C: Activities to try in the classroom to enhance understanding of figurative numbers and their relationships

You have been studying some figurative numbers. You might have discovered that the topic is inexhaustible and that many more questions could be looked at. Figurative numbers, and number patterns in general, are a rich ground for investigations. Here follows your assignment. You are asked to try out some of the activities in the classroom. The activities suggested are described below your assignment.

Unit 3: Practice activity

1. Try out the 'handshake' activity in the classroom (Worksheet 1).
2. Write an evaluation of the lesson in which you presented the activity. Some questions you might want to answer could be: What were the strengths and weaknesses? What needs improvement? How was the reaction of the pupils? What did you learn as a teacher from the lesson? Could all pupils participate? Were your objective(s) attained? Was the timing correct? What did you find out about pupils' investigative abilities? What further activities are you planning to strengthen pupils' problem solving and investigative work? Were you satisfied with the outcome of the activity?
3. Use the information on figurative numbers presented in the previous section B to develop investigative activities for your class. Evaluate the activity.

Present your assignment to your supervisor or study group for discussion.

Pupils activity: Handshakes

Objectives of the activity:

- to enhance problem solving strategies
- to use acting out as a strategy to collect data
- to relate a 'real life' situation to triangular numbers

To the teacher:

The triangular numbers appear in the 'handshake' activity. As we do not expect pupils to be able to generate rules for the *n*th term of a sequence that is not linear, assist pupils to find the general rule. For some groups of pupils, continuing the pattern might be enough.

The pupils predict, collect data and compare their predictions with the outcomes. The introduction to the class can be short. The predicted outcomes are placed on the board.

Pupils work in groups of 4 – 6 counting the number of handshakes by forming sub-groups of 2, 3, 4, .. pupils. Groups report back on different strategies used in solving the problem: starting with simpler case, tabulating data, looking for patterns.

The completed table will look as follows

Number of people	2	3	4	5	6	7			n
Number of handshakes	1	3	6	10	15	21			$\frac{1}{2}n(n-1)$

Explanations considered might be:

i) If there are 3 people the first person will shake hands with 2 people (and sit down), the next person will shake hands with 1 person.

Total: $1 + 2 = 3 = \frac{1}{2}(2 \times 3)$

For four persons: $1 + 2 + 3 = 6 = \frac{1}{2}(3 \times 4)$

For five persons $1 + 2 + 3 + 4 = 10 = \frac{1}{2}(4 \times 5)$

The pattern generalizes to:

For n persons $1 + 2 + 3 + ... + (n-1) = \frac{1}{2}[(n-1) \times n]$

(ii) One handshake is exchanged between 2 people. If there are 10 people each will shake hands with nine others: 10×9, but now the handshakes have been counted double—A with B and B with A have been counted separately—so we must take half of the number: $\frac{1}{2} \times 10 \times 9$.

This generalizes to $\frac{1}{2} \times n \times (n-1)$.

Summary

Polygonal numbers provide a highly visual aid to learning many concepts of sequences and other algebra. They also provide easily managed settings for problem solving skills to be learned and used, both singly and in groups. You should find that in every class where you teach using polygonal patterns and numbers, more students grasp algebraic concepts more rapidly than you otherwise would expect.

Problem.

Suppose each of you in class shakes the hand of each other, once. How many handshakes are there in all?

Guess: ____

Should we do it and keep count of all handshakes? Could you remember whose hand you had already shaken?

Answer: ____

Should we write out each possibility, then count them?

Answer: ____

Form groups of 2 pupils, 3 pupils, 4 pupils, etc. and count the number of handshakes if all pupils in a group shakes the hand of each other, once.

Record your results in a table.

Number of people	2	3	4	5	6	7			n
Number of handshakes									

Let's go back to the first question of the worksheet:

Suppose each of you in class shakes the hand of each other, once. How many handshakes are there in all?

Answer: ____

Describe how you found your answer.

____

____

How was your guess? Overestimated? Underestimated? Can you explain the difference?

____

Challenge: Can you find the number of handshakes for *n* people?

Unit 3: Answers to self mark exercises

Self mark exercise 1

1. 28, 36, 45

3. add 2, add 3, add 4, .. to the previous term

4. $t_n = \frac{1}{2} n(n + 1)$

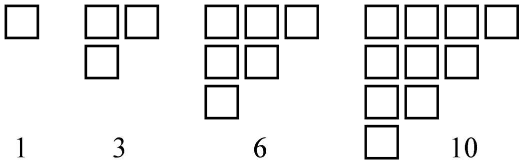
Self mark exercise 2 & 3

Answer to question can be obtained from the table

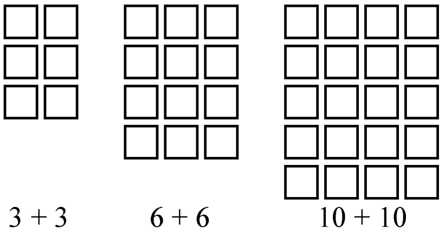
Type of number	**Number of sides**	**First terms in the sequence**	***n*th term: t_n**
triangular	p = 3	1, 3, 6, 10, 15, 21, 28, ...	$\frac{1}{2} n\ (n + 1)$
square	p = 4	1, 4, 9, 16, 25, 36, 49, ...	$n^2 = \frac{1}{2} n\ (2n + 0)$
pentagonal	p = 5	1, 5, 12, 22, 35, 51, 70,...	$\frac{1}{2} n\ (3n - 1)$
hexagonal	p = 6	1, 6, 15, 28, 45, 66, 91, ...	$\frac{1}{2} n\ (4n - 2) = n\ (2n - 1)$
heptagonal	p = 7	1, 7, 18, 34, 55, 81, 112, ...	$\frac{1}{2} n\ (5n - 3)$
octagonal	p = 8	1, 8, 21, 40, 65, 96, 133, ...	$\frac{1}{2} n\ (6n - 4) = n(3n - 2)$
nonagonal	p = 9	1, 9, 24, 46, 75, 111, 154, ...	$\frac{1}{2} n(7n - 5)$
decagonal	p = 10	1, 10, 27, 52, 85, 126, 175, ...	$\frac{1}{2} n(8n - 6) = n(4n - 3)$
p-gonal	p	1, p, $3p - 3$, $6p - 8$, $10p - 15$	$\frac{1}{2} n[(p - 2)n - p + 4]$

Self mark exercise 4

1. The nth triangular number is $\frac{1}{2}\ n\ (n + 1)$.



Triangular numbers



Double a triangular number to give a rectangular number

Algebraic proof

$2 \times$ triangular number $= 2 \times \frac{1}{2}\ n\ (n + 1) = n(n + 1)$. $n(n + 1)$ is a rectangular number.

2. $t_n = \frac{1}{2} n(10n - 8)$ is the nth dodecagonal number.

3. $p = 8$ or $p = 40$

4. 1, 5, 12, 22, 35, 51, 70, ... are the pentagonal numbers

   1, 1 + 4, 3 + 9, 6 + 16, 10 + 25, 15 + 36, 21 + 49, .. the pentagonal numbers written as sum of a triangular number (1, 3, 6, 10, 15, 21, ...) and a square number (4, 9, 16, 25, 36, 49, ...)

   Drawing of patterns is left to you.

   Algebraic proof:

   $p_n = \frac{1}{2} n(3n - 1) = \frac{1}{2} n[n - 1 + 2n] = \frac{1}{2} n(n - 1) + \frac{1}{2} n(2n) = t_{n-1} + s_n$ (the $(n - 1)$st triangular number plus the nth square number).

5b. nth hex number $3n^2 - 3n + 1$

6b. nth central number $2n^2 - 2n + 1$

7. 1, 5, 13, 25, 41, ... are the central numbers.

   You can write them as 1, 1 + 4, 4 + 9, 9 + 16, 16 + 25, .. i.e., sum of two square numbers.

   Algebraic proof:

   $2n^2 - 2n + 1 = n^2 + n^2 - 2n + 1 = n^2 + (n - 1)^2$ being the sum of the nth and $(n - 1)$st square number.

Unit 4: Rational and irrational numbers

Introduction to Unit 4

Counting and using numerals to represent numbers most likely belong to some of the earliest activities of mankind. When we speak of our numeration system we may be referring to either of two distinct ideas: the written symbols or the number words (at times accompanied by gestures) (Zaslavvsky, 1973). The structure of the number system we use now developed through the ages and each new discovery to extend the system initially met with great resistance; e.g., irrational numbers, negative numbers and complex numbers all were initially unacceptable to ‘mainstream’ mathematics. This unit looks at part of the structure of the number system.

Purpose of Unit 4

The aim of this unit is to:

- review and extend your knowledge on the structure of the number system in order give you more confidence in teaching of numbers.

Objectives

When you have completed this unit you should be able to:

- explain the structure of the number system (whole numbers, integers, rational numbers, irrational numbers, real numbers, complex numbers)
- explain the difference between rational and irrational numbers, illustrating with examples
- explain and illustrate the denseness of rational and irrational numbers on the number line
- explain why the integers are not dense on the number line
- change recurring decimal fractions to rational numbers
- set and justify activities for classroom use to enhance understanding of the number system

Time

To study this unit will take you about 4 hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topic.

Unit 4: Rational and irrational numbers

Section A: The growing number system

The classification of numbers and the structure of the number system are far from straightforward for pupils. The *natural* or *counting* numbers will be accepted by pupils as a natural abstraction of the child's experience with counting of objects and ordering them. The extensions of the system are far less straightforward and form a conceptual leap. Negative integers, rational numbers, irrational numbers and complex numbers were created by people to describe new situations.

At primary school pupils are introduced to the counting numbers 1, 2, 3, 4, ... Next the 0 (no members in the set) is added. Historically the introduction of a numeral to represent the absence of certain values in a place value system was a great step forward. It dates back to the second century BC (India) and was introduced to the West through the Arabic mathematician Al-Khwarizmi (AD 680). Babylonian, Egyptian and Roman numerals have no symbol for zero and are much harder to calculate with.

Later, rational numbers (referred to as fractions) were introduced as part of a whole: a whole is divided into equal parts, each part is a fraction of the whole. Decimals (finite such as 4.56, 0.08934 or recurring such as 0.33333..., 1.27272727...) and percents are not extending the number system of the rational numbers; they are different ways of representing rational numbers.

For example: $\frac{2}{5} = 0.4 = 40\%$ or $\frac{1}{3} = 0.3333... = 33\frac{1}{3}\%$

Showing the need to extend the number line with the negative integers (and rational numbers) is readily accepted (to make it possible to solve equations such as $2x + 5 = 1$ and $2x + 3 = 2$). Historically, negative numbers were greatly resisted as 'not real'. What does $^{-}3$ apples mean? Nowadays pupils are used to temperature scales including negative numbers (temperature below zero on a Celsius scale).

To introduce irrational numbers, explaining that these cannot be expressed as rational numbers, is too difficult for most children in the age range 12 - 16. The discovery of irrational numbers is credited to Pythagoras who found that the diagonal of a square is not a rational multiple of its side. For example $\sqrt{2}$ cannot be expressed as a rational number $\frac{p}{q}$ (with p and q integers and $q \neq 0$), its decimal expansion does not terminate or become periodic (recurring).

At a higher level, algebra led to the introduction of complex numbers, in order to find a solution to equations such as $x^2 = {}^{-}1$. The number $i = \sqrt{^{-}1}$ was invented to deal with this situation. The square root of negative numbers are called ' imaginary', however they are just as real as any other number invented by the human mind. In the physical sense, $\sqrt{2}$ or π are not real either, as we cannot measure with an infinite precision. Complex numbers are generally introduced after the age of 16 as it requires a fair level of abstract thinking.

Section B: Natural numbers, counting numbers and whole numbers

Conventions differ as to what are considered to be natural numbers. Some authors take the natural numbers to be the numbers represented by the numerals 1, 2, 3, 4, , ... and the whole numbers as 0, 1, 2, ... Other authors do not distinguish between the whole numbers with 0 and without 0. In this unit we will use the convention to denote the natural numbers (or counting numbers) by N and take them to be 1, 2, 3, 4,... . The whole numbers will be indicated with W and be the natural numbers N and 0;
i.e., W = {0, 1, 2, 3, 4, ...}

Section C: The number system: notation convention used

The following notation will be used throughout,

N, the **natural numbers**, counting numbers or positive integers: 1, 2, 3, 4, ...

W, the **whole numbers** or non negative integers: 0, 1, 2, 3, 4, ...

Z the **integers** ..... $^{-}4$, $^{-}3$, $^{-}2$, $^{-}1$, 0, 1, 2, 3, ... The Z is from the German word for numbers: Zahlen.

$\Re$ the **real numbers**, rational numbers together with the irrational numbers.

Q, the **rational numbers**, all the numbers that can be written in the form $\frac{p}{q}$, with p and q integers and $q \neq 0$. The Q is from quotient.

Irrational numbers are numbers that *cannot* be written as rational numbers; their decimal format leads to an infinite non recurring decimal fraction.

Examples of irrational numbers are

- roots that cannot be expressed as a finite or recurring decimal:
  $\sqrt{5},\ \sqrt{\frac{5}{7}},\ 5+\sqrt{3},\ 6\sqrt{11},\ \sqrt[3]{2}$
- numerals involving π:
  π, $3\pi + 2$, $\sqrt{\pi}$
- many trigonometric values:
  sin 20°, cos 45°
- non-terminating and non-recurring decimals:
  0.12112111211112111112....
  0.123456789101112131415161718192021....

Some numerals look like irrationals but are NOT!

For example the following are NOT irrational numbers:

$\sqrt{121}$ which is a representation of the rational number 11.

$\sqrt{12} \times \sqrt{3}$ which is representing the rational number 6.

sin π a representation of the rational number 0.

$2\sqrt{24} \div \sqrt{6}$ representing the rational number 4.

Recurring decimal fractions are rational numbers. Here you can see how to change a recurring fraction to its rational format.

Let us look at the recurring decimal 0.1212121212 .....

Let us call it rational format x, them x = 0.1212121212 .....

Because the recurring part consists of two digits we multiply by 100 and we obtain

$$100x = 12.1212121212....$$

We had $x = 0.1212121212 .....$

Subtracting, $99x = 12$. Hence $x = \frac{12}{99} = \frac{4}{33}$

Here is another example

2.213434343434...

Use the same procedure as above. Note that the recurring digits 34 are preceded by a non-recurring part 2.21; we therefore multiply by 10 000.

if $x = 2.213434343434$ ..

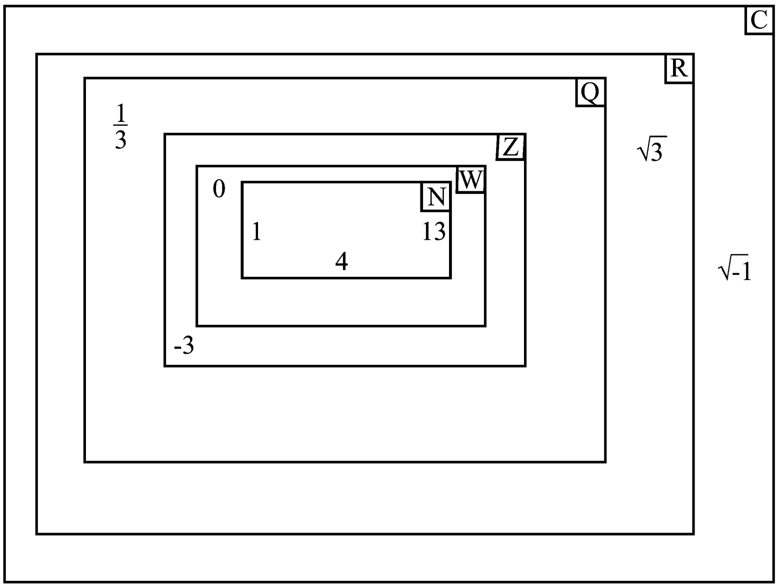
then $10\,000\,x = 22134.34343434$ ..

Subtracting, $10\,000\,x - x = 22132.13$

$$9999x = \frac{2213213}{100} \qquad x = \frac{2213213}{999900}$$

C, the **complex numbers**, are any numbers that can be expressed in the form $a + ib$, where a and b are real numbers and $i = \sqrt{-1}$. a is called the real part of the complex number and ib the imaginary part. For example $3 - 2i$, $4i$, $3\ (= 3 + 0i)$, $0.5\ (= 0.5 + 0i)$ are examples of complex numbers.

The structure of the number system can be illustrated with the following labelled Venn diagram. A typical example of each type of number is given in the diagram.



Note the inclusive nature of the structure: all integers are rational numbers as $^{+}4$ and $^{-}3$ **can** be expressed as a rational number. For example

$^{+}4 = \frac{4}{1} = \frac{12}{3} = \frac{^{-}16}{^{-}4} = \ldots$ and $^{-}3 = \frac{3}{^{-}1} = \frac{^{-}27}{9} = \ldots$

At a higher level: whole numbers, rational numbers, real numbers are also complex numbers with the imaginary part equal to 0.

Section D: Denseness of the numbers

Between two consecutive whole numbers, for example 11 and 12, NO other whole number can be found. This is expressed by saying: the whole numbers are NOT dense on the number line. If you represent the whole numbers by points on the number line there are ‘gaps’, sections not filled with numbers.

The rational numbers on the other hand are dense on the number line: between any two rational numbers you can always find infinitely many other rational numbers (for example the average of the two rational numbers will be between them).

Between $\frac{11}{13}$ and $\frac{12}{13}$ you can find the rational numbers $\frac{23}{26}, \frac{34}{39}, \frac{59}{65}$, ... and infinitely more.

By now we could think that the number line is ‘full’, no gaps left to place any other number. However there are still the irrationals! They are also dense on the number line: between any two rational numbers infinitely many other irrationals can be found and also between any two irrational numbers infinitely many other irrational numbers can be found.

For example between $\frac{11}{13}$ and $\frac{12}{13}$ you can find the irrational numbers $\frac{1}{10}\sqrt{73}$, $\frac{1}{2}\sqrt{3}$, $\frac{1}{10}\sqrt{19}$, ...

Check this using your calculator.

Between $\sqrt{17}$ and $\sqrt{18} = 3\sqrt{2}$ you can find the rational numbers 4.13, 4.15, 4.152, ... and the irrational numbers $\frac{1}{2}\sqrt{70}$, $\frac{1}{2}\sqrt{71}$, ... Check this using your calculator.

Cantor, a German mathematician living at the beginning of the 20th century (he died in 1918), was able to prove that the vast majority of numbers are irrational. The rational numbers are only a small collection of numbers compared to the irrational numbers. He expressed this by introducing different forms of 'infinity'. The rational numbers are 'countably infinite' while the irrational numbers are 'uncountably infinite' as their order of infinity is higher.

Section E: Proving that $\sqrt{2}$ is irrational

Remember that in self mark exercise 3 you showed that if n^2 is an even number, then also n is even. This we will use in the following proof **by contradiction.**

A proof by contradiction starts by assuming that what is to be proved is **not** true. In this case you assume that $\sqrt{2}$ is a rational number. From that assumption you proceed to come to a statement that contradicts the assumption. The conclusion is then that the assumption ($\sqrt{2}$ being rational) cannot be true, so that $\sqrt{2}$ must be irrational.

Carefully go over the following proof ensuring every step is clear and understood. It is rather abstract algebraic reasoning (not appropriate for pupils in the age range 12 - 14, with exception of the very high achievers). You might have to read this section more than once.

Assume $\sqrt{2}$ is rational so that it can be written as $\frac{m}{n}$ where m and n are whole numbers with no common factor other than 1. That is to say that $\frac{m}{n}$ is in its simplest form and cannot be simplified.

If $\sqrt{2} = \frac{m}{n}$

then $2 = \frac{m^2}{n^2}$ (squaring both sides)

$2n^2 = m^2$ (multiplying both sides by n^2)

This tells you that $2n^2$ is divisible by 2, as it starts by a factor 2. So also m^2, which is equal to $2n^2$ must be divisible by 2 and be an even number.

Since m^2 is even so also is m. (This you have proved in Unit 1, Section D.)

So it must be possible to write $m = 2p$ for a certain whole number p.

Substituting $m = 2p$ gives

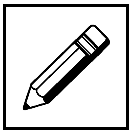
$2n^2 = (2p)^2$

$2n^2 = 4p^2$ (expanding)

$n^2 = 2p^2$ (dividing both sides by 2)

Now the same argument applies again: Since n^2 is even (as $2p^2$ is even) so must be n.

You have now deduced that both m and n are even numbers. Both m and n, being even, have a factor 2. This contradicts the assumption that m and n had only 1 as common factor. Our assumption ($\sqrt{2}$ being rational) must have been incorrect. Hence $\sqrt{2}$ is NOT rational, that is $\sqrt{2}$ is irrational.



Self mark exercise 1

1. Are 'positive integers' and 'non-negative integers' referring to the same set of numbers?
2. Why is it stated in the definition of the rational numbers $\frac{p}{q}$, that $q \neq 0$?
3. What is the difference between fractions and rational numbers? Illustrate with examples.
4. Do you know numbers that are NOT rational? What are they called? Illustrate with examples.
5. How will you classify the percents?
6. To what number system(s) do the decimals belong?
7. Are the integers dense on the number line? Explain.
8. Classify the following numerals as integers, rational numbers, irrational numbers and/or real numbers: $^-2.1$, $\sqrt{25}$, $\frac{6}{2}$, $\frac{\pi+1}{\pi}$, $\sqrt{\pi^2}$, sin 90°, cos 45° ÷ $\sqrt{2}$, $\sqrt{2.25}$, 0, $1.\underline{3}$ (recurring fraction), 0.123123123....., $(\frac{1}{4}\sqrt{6})^2$
9. Find three numbers between 3 and $3\frac{1}{2}$ which are (i) rational (ii) irrational.
10. Find three numbers between $\sqrt{10}$ and $\sqrt{11}$ which are (i) rational (ii) irrational.

Self mark exercise 1 continued on next page

Self mark exercise 1 continued

11. p is a positive integer such that $\sqrt{p} = 23.4$ (1 decimal point). Find the value(s) of p and explain why $\sqrt{p}$ is irrational.
12. Investigate whether the following statements are true or false, justifying your answers, illustrating with examples and non examples.
    a. The sum of any two irrational numbers is an irrational number.
    b. The sum of any rational and any irrational number is an irrational number.
    c. The product of any two irrational numbers is an irrational number.
    d. The product of any rational and any irrational numbers is an irrational number.
13. Write the following recurring decimals as fractions: $0.\underline{7}$, $0.\underline{34}$, $0.2\underline{5}$, $0.3\underline{41}$, $0.\underline{9}$
14. Prove that $\sqrt{3}$ is an irrational number.
15. Is $\sqrt{4}$ irrational? What happens if you attempt to prove by contradiction that $\sqrt{4}$ is irrational?
16. Give two examples of irrational numbers p and q (different from each other) such that $\frac{p}{q}$ is a rational number.
17. State for the following types of numbers whether they are **always rational, possibly rational** or **never rational.** Justify your answer.
    a. finite decimals
    b. infinite decimals
    c. square root of whole numbers
18. A pupil in an attempt to define an 'irrational number' said "An irrational number is a number which, in its decimal form, goes on and on."
    a. What example would you present to the pupil to show that the definition is not correct?
    b. How is the definition to be 'refined' to make it correct?
19. π is an irrational number. Various rational approximations to π are in use. Using your calculator approximation of π find the percent error when using the following historical rational approximations.

    (i) $\frac{22}{7}$ (ii) $3\frac{1}{8}$ (iii) $\sqrt{10}$

Check your answers at the end of this unit.

Section F: Rational and irrational numbers in the classroom

What do you want pupils in the age range 12 - 14 to learn about the number system? In the sections above: what was new to you? What in the above section could be used in the classroom for all / some pupils? What do you consider to be definitely inappropriate to present to pupils and why? ("It is difficult" is generally NOT a good argument as many pupils can handle 'difficult' concepts provided they are presented at their level with sufficient guidance from the teacher as facilitator.)

Reflection

1. Write down on a piece of paper what you consider can be learned by pupils in the age range 12 - 14 about the number system.
2. For each of the concepts listed in the above question 1, suggest a teaching method that can be used to assist pupils' learning.

Did you include the following?

1. Given any number, pupils should be able to identify it as belonging to the natural numbers, integers, rational numbers, irrational numbers.
2. Pupils should be able to give examples and non-examples of numbers belonging to the natural numbers, integers, rational numbers, irrational numbers.
3. Pupils should be able to relate the rational numbers to either terminating decimal numbers or non-terminating recurring decimal numbers.
4. Pupils should be able to find rational / irrational numbers between two given rational / irrational numbers using a calculator.
5. Pupils should be able to explain that the rational numbers are "dense on the number line": between any two rational numbers there exist infinitely many other rational numbers.
6. Pupils should be able to identify non-terminating, non-repeating decimals with irrational numbers and be able to give examples.
7. Pupils should be aware of the 'inclusive' nature of the number system e.g., all integers are rational numbers, all natural numbers are integers, etc.
8. Pupils should be able to state the rational equivalent of some simple recurring decimal fractions, e.g., $0.\underline{3} = \frac{1}{3}$, $0.\underline{6} = \frac{2}{3}$, $0.\underline{1} = \frac{1}{9}$.
9. Pupils should be able to give π as an example of an irrational number and show awareness of 3.14, $\frac{22}{7}$ being approximations to π by appropriate use of $\approx$ symbol.

Some suggestions for pupils activities

1. Investigate rational numbers and terminating decimals. Which rational numbers have a terminating decimal format? Expected outcome: all fractions with denominators, when factorised, of the form $2^p \times 5^q$ p and q being whole numbers (0, 1, 2, ...)
2. Investigate patterns in recurring decimals e.g., the recurring decimal format for $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}$ and $\frac{6}{7}$. Expected outcome: pupils discover that in all decimal formats the very same digits (142857) appear but start at different digits.
3. Investigate non-terminating, non-recurring decimals. For example:

   0.79779777977779777779777777779...

   How many digits precede the one hundredth 9?

   0.12112211122211112222...

   Among the first 1000 digits how many are 1, how many 2?

   Expected outcome: enhancing understanding of non-terminating, non-recurring decimals and enhancing of problem solving strategies.

Unit 4, Assignment 1

1. Using the objectives for pupils, the ideas you learned yourself in this unit and the suggestions above develop an activity based lesson on (i) number systems: classification of numbers and (ii) irrational numbers. Pupils should be able, in their groups, to learn about number systems and irrational numbers.
2. Try out your lessons with your pupils.
3. Write an evaluation of the lessons. Some questions you might want to answer could be: What were the strengths and weaknesses? What needs improvement? How was the reaction of the pupils? What did your learn as a teacher from the lesson? Could all pupils participate? Were your objective(s) attained? Was the timing correct? What did you find out about pupils' investigative abilities? What further activities are you planning to strengthen pupils' understanding of the number system and more especially about the nature of irrational numbers. Were you satisfied with the outcome of the activity?

Present your assignment to your supervisor or study group for discussion.

Summary

You have come to the end of the first module. It is expected that you have reviewed and increased your knowledge of number systems, classes of numbers, number sequences and the multiple relationships among numbers. Apart from having increased your own knowledge, you should have worked with teaching methods that might not have been part of your practice before you started with this module. The experience in the classroom with a pupil centred approach using, among others, games, challenging questions and problem solving / investigation activities should have widened your classroom practice and methods. The task of a teacher is in the first place to create an environment for the pupils in which they can learn by **doing** mathematics. Your final module assignment is to assess the progress you have made.

Module 1, Assignment

1. Read again what you wrote at the start of this module "My teaching approach and methods". If you were to write now again on this would there be any changes? Justify the changes in your teaching approach and practice or justify why there is no need for you to make any changes.

2. Pupils learn by doing mathematics (practical activities, problem solving, investigations) and discussion with each other and the teacher on how they work the tasks set. Design a sequence of lessons on "Numbers" to enhance problem solving skills in your pupils. Write out the lesson plans and worksheets, taking into account the wide range of achievement levels of your pupils in your class. Each and every pupil should be able to develop problem solving skills at their own level.

3. Write an evaluation of the lessons. Some questions you might want to answer could be: What were the strengths and weaknesses? What needs improvement? How was the reaction of the pupils? What did you learn as a teacher from the lessons? Could all pupils participate? Were your objective(s) attained? Was the timing correct? What did you find out about pupils' investigative abilities? What further activities are you planning to strengthen pupils' problem solving skills?

4. Investigate: How many factors do the different classes of numbers you encountered in this module have (square numbers, cubes, figurative numbers, ...)? Which numbers have a square number of factors? Which numbers have a prime number of factors?

Present your assignment to your supervisor or study group for discussion.

Unit 4: Answers to self mark exercises

Self mark exercise 1

1. Positive integers are 1, 2, 3, 4, 5, ...

   Non-negative integers are 0, 1, 2, 3, 4, 5, ...
2. Dividing by 0 is undefined.
3. Rational numbers are numbers that can be expressed in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

   Fractions are numbers that can be expressed in the form $\frac{a}{b}$ where a and b are real numbers and $b \neq 0$.

   All rational numbers are fractions (as the real numbers include the rational numbers) but not all fractions are rational numbers. For example, $\frac{\pi}{2}$, $\frac{\sqrt{2}}{2}$ are fractions but NOT rational numbers.
4. (i) irrational numbers such as $\sqrt{3}$, e, π, sin 60° (ii) complex numbers such as $\sqrt{-2}$
5. Percents are rational numbers. % represents the denominator 100.
6. Terminating and recurring decimals are rational numbers, non-terminating and non-recurring decimals are irrational. See for example the text.
7. No, as between two consecutive integers you cannot find another integer.
8. Rational are:

   $^{-}2.1 = -\frac{21}{10}$, $\sqrt{2.25} = 1.5$, $\cos 45° \div \sqrt{2} = \frac{1}{2}$, $1.\underline{3} = 1\frac{1}{3}$,

   $0.123123123..... = \frac{41}{333}$, $(\frac{1}{4}\sqrt{6})^2 = \frac{3}{8}$

Integers are: $\sqrt{25} = 5$, $\frac{6}{2} = 3$, sin 90° = 1, 0 integer

   Real / irrational are

   $\frac{\pi + 1}{\pi}$ $\sqrt{\pi^2}$

   N.B. integers are also rational numbers and real numbers; rational numbers are also real numbers.

9. For example: (i) Rational $3\frac{1}{10}$, $3\frac{1}{5}$, $3\frac{1}{3}$, $3\frac{1}{4}$ (ii) Irrational $\sqrt{10}$, $\sqrt{11}$, 3π

10. (i) Rational 3.2, 3.24, 3.3 (ii) Irrational $\frac{1}{2}\sqrt{42}$, $\frac{1}{3}\sqrt{95}$, $\sqrt[3]{1020}$

11. 546, 547, 548 or 549, none of these being a perfect square.

12 a. not true, e.g., $\sqrt{2} + (^-\sqrt{2}) = 0$, if restricted to positive irrationals it is a true statement.

b. true, e.g., $\frac{1}{2} + \sqrt{2}$ is irrational

c. not true, e.g., $\sqrt{2} \times \sqrt{2} = 2$, $\sqrt{3} \times \sqrt{12} = 6$

d. not true, e.g., $\frac{0}{2} \times \sqrt{2} = 0$, if 0 is excluded from the rational numbers the statement is true.

13. $0.\underline{7} = \frac{7}{9}$, $0.\underline{34} = \frac{34}{99}$, $0.2\underline{5} = \frac{23}{90}$, $0,3\underline{41} = \frac{169}{495}$, $0.\underline{9} = 1$

14. Assume $\sqrt{3}$ is rational so that it can be written as $\frac{m}{n}$ where m and n are whole numbers with no common factor other than 1. That is to say that $\frac{m}{n}$ is in its simplest form and cannot be simplified.

If $\sqrt{3} = \frac{m}{n}$

then $3 = \frac{m^2}{n^2}$ (squaring both sides)

$3n^2 = m^2$ (multiplying both sides by n^2)

This tells you that $3n^2$ is divisible by 3, as it starts by a factor 3. So also m^2, which is equal to $3n^2$ must be divisible by 3.

Since m^2 is divisible by 3 so also is m.

So it must be possible to write $m = 3p$ for a certain whole number p.

Substituting $m = 3p$ gives

$3n^2 = (3p)^2$

$3n^2 = 9p^2$ (expanding)

$n^2 = 3p^2$ (dividing both sides by 3)

Now the same argument applies again: Since n^2 is divisible by 3 (as $3p^2$ is divisible by 3) so must be n.

You have now deduced that both m and n are divisible by 3. Both m and n, being divisible by 3, have a factor 3 in common. This contradicts the assumption that m and n had only as common factor 1. Our assumption ($\sqrt{3}$ being rational) must have been incorrect. Hence $\sqrt{3}$ is NOT rational, that is $\sqrt{3}$ is irrational.

15. Assume $\sqrt{4}$ is rational so that it can be written as $\frac{m}{n}$ where *m* and *n* are whole numbers with no common factor other than 1. That is to say that $\frac{m}{n}$ is in its simplest form and cannot be simplified.

If $\sqrt{4} = \frac{m}{n}$

Then $4 = \frac{m^2}{n^2}$ (squaring both sides)

$4n^2 = m^2$ (multiplying both sides by n^2)

This tells you that $4n^2$ is divisible by 4, as it starts with a factor 4. So also m^2, which is equal to $4n^2$ must be divisible by 4.

At this point the proof breaks down as we **cannot** conclude:

Since m^2 is divisible by 4 so also is *m*, because *m* could be even and then m^2 has a factor 4. We can only conclude that *m* is even—NOT necessarily a multiple of 4.

If you would continue now:

So it must be possible to write $m = 2p$ for a certain whole number *p*.

Substituting $m = 2p$ gives

$4n^2 = (2p)^2$

$4n^2 = 4p^2$ (expanding)

$n^2 = p^2$ (dividing both sides by 4)

Which does not lead to a contradiction.

16. $\frac{2\sqrt{12}}{3\sqrt{3}} = \frac{4}{3}, \frac{2\pi}{3\pi} = \frac{2}{3}$

17 a. always rational

b. possibly rational (if recurring)

c. possibly rational, e.g., $\sqrt{4}$

18 a. $0.333... = \frac{1}{3}$

b. to be added: 'and is not recurring'

19. (i) 0.04% (2 dp) (below)

(ii) 0.53% (2 dp) (above)

(iii) 0.66% (2 dp) (below)

References

Cockcroft, W.H. (1982) *Mathematics Counts.* London HMSO.

Polya , G. (1957) *How To Solve It. A New Aspect of Mathematical Aspect.* 2nd Edition. Princeton University Press. ISBN 0691 0235 65

Zaslavsky, C. (1979) *Africa Counts*. Lawrence Hill Books. ISBN 1556 5207 51

Additional References

The following books have been used in developing this module and contain more ideas for the classroom.

UB-Inset, University of Botswana, *Patterns and Sequences*, 1997

NCTM, *Developing Number Sense*, 1991, ISBN 087 353 3224

NCTM, *Patterns and functions*, 1991, ISBN 087 353 3240

ATM, *Numbers Everywhere*, 1972, ISBN 090 009 5172

Kirby D, *Games in the teaching and learning of mathematics*, 1992, ISBN 052 142 3201

NCTM, *Understanding rational numbers and properties*, 1994, ISBN 087 353 3259

NCC, *Mathematics programmes of study*, 1992, ISBN 187 267 6898

NCTM, *Teaching and learning of algorithms in school mathematics*, 1998, ISBN 087 353 4409

London, R., *Nonroutine Problems: Doing Mathematics*, 1989, ISBN 093 976 5306

Mathematical Challenges, Jason Publications, 1989.

Spotline on Understanding: Strategies for solving problems, Jason Publications, 1996, ISBN 093 976 65691

Further Reading

The following series of books is highly recommended to use with the text in this module. The Maths in Action Books (book 3, book 4 and book 5 to be published in 2000) with the accompanying Teacher's Files are based on a constructive approach to teaching and learning. The student books allow differentiation within the classroom. The books are activity based: learning by doing and discovery. The Teacher's File contains materials which may be photocopied: worksheets, games and additional exercises for students.

OUP/Educational Book Service, *Maths in Action Book 1*
ISBN 019 571776 7, P92

OUP/EducationalBook Service, *Maths in Action Book 2*
ISBN 019 ... (published 1999)

OUP/Educational Book Service, *Maths in Action Teacher's File Book 1*
ISBN 019 ...(published 1999)

OUP/Educational Book Service, *Maths in Action Teacher's File Book 2*
ISBN 019 … (published 1999)

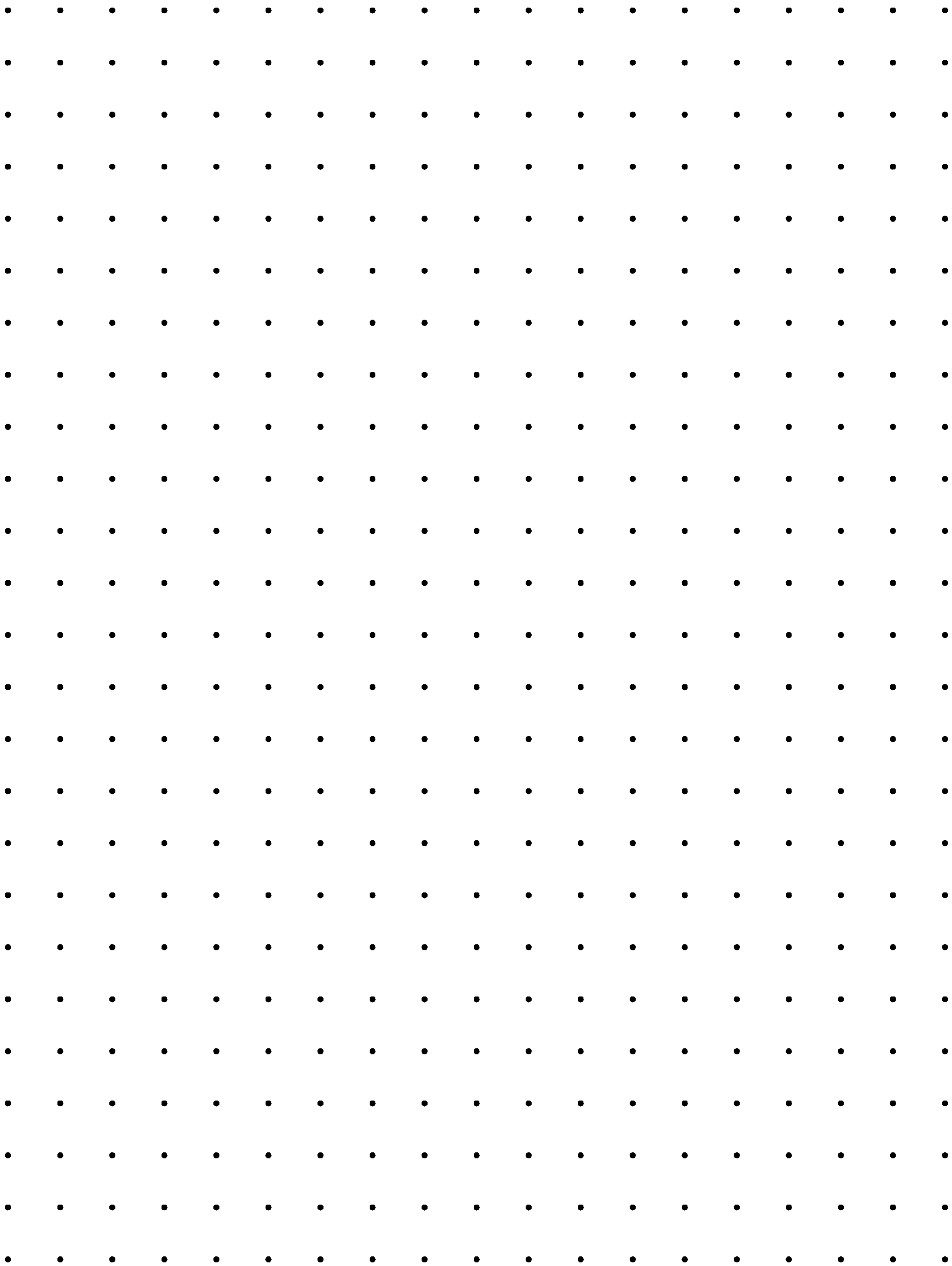
G Glossary

Base	in n^p, n is called the base
Complex numbers	any number that can be expressed as $a + bi$ where a and b are real and $i = \sqrt{-1}$
Consecutive numbers	two whole numbers following each other, e.g., n and $n + 1$ are two consecutive whole numbers, $2n$ and $2n + 2$ are two consecutive even numbers
Counting numbers	see *whole numbers*
Cube numbers	natural numbers that can be represented by a cube pattern of unit cubes; numbers of the form n^3
Directed numbers	see *integers*
Factors	factors of a natural number N are natural numbers that divide into N
Figurative numbers	natural numbers that can be represented in a geometric dot pattern e.g., a triangle, square, rectangle, pentagon, etc.
Fraction	any number that can be expressed as $\frac{p}{q}$, where p and q are real and $q \neq 0$
Index	in the expression n^p, p is the index of the power n^p
Integers	the positive, negative and 0 whole numbers
Irrational numbers	numbers that are not rational
Linear expression	expressions of the form $ax + b$
Multiples	multiples of a natural number N are the numbers pN where p is a natural number
Natural numbers	1, 2, 3, 4, ... Also called the Counting numbers, they do not include zero or the negative numbers.
Number sense	a feel for size of numbers and having referent objects related to numbers, being aware of multiple relationships among numbers and the effect of operations on numbers
Numeral	representation of a number
Oblong numbers	rectangular numbers that are not square numbers
Polynomial numbers	natural numbers that can be represented in a polygonal dot pattern

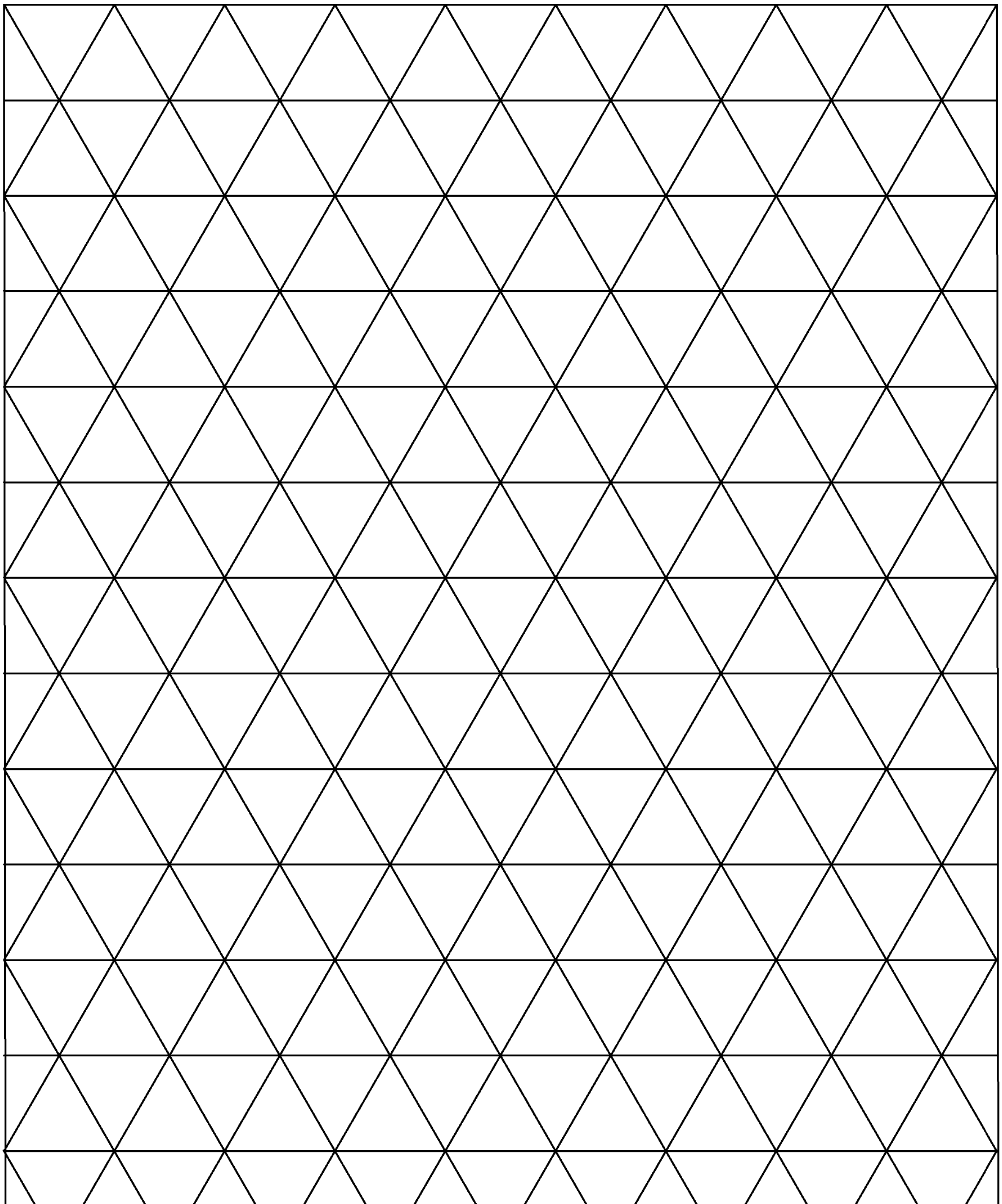
Power	expression of the form n^p. If p is natural number it is the product p factors n.
Quadratic expression	expression of the form $ax^2 + bx + c$
Rational numbers	any number that can be expressed as $\frac{p}{q}$ (with p and q integers and $q \neq 0$)
Real numbers	set of rational and irrational numbers
Recurring decimal	decimal in which one or a string of digits keeps on repeating e.g., 0.3333..., 0.123 434 343 4 ...
Rectangular numbers	natural numbers with at least three factors and hence can be represented in a rectangular dot pattern
Square numbers	natural numbers that can be represented by a square dot pattern 1, 4, 9, 16, ...
Triangular numbers	natural numbers that can be represented by triangular dot patterns 1, 3, 6, 10, ...
Whole numbers	the numbers 0, 1, 2, 3, 4, ...

Appendix 1

Appendix 2



Appendix 3



Appendix 4