Module 5

Junior Secondary Mathematics

Solving Equations

THE COMMONWEALTH *of* LEARNING

Science, Technology and Mathematics Modules

for Upper Primary and Junior Secondary School Teachers of Science, Technology and Mathematics by Distance in the Southern African Development Community (SADC)

Developed by
The Southern African Development Community (SADC)

Ministries of Education in:

- **Botswana**
- **Malawi**
- **Mozambique**
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- **South Africa**
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SCIENCE, TECHNOLOGY AND MATHEMATICS MODULES

This module is one of a series prepared under the auspices of the participating Southern African Development Community (SADC) and The Commonwealth of Learning as part of the Training of Upper Primary and Junior Secondary Science, Technology and Mathematics Teachers in Africa by Distance. These modules enable teachers to enhance their professional skills through distance and open learning. Many individuals and groups have been involved in writing and producing these modules. We trust that they will benefit not only the teachers who use them, but also, ultimately, their students and the communities and nations in which they live.

The twenty-eight Science, Technology and Mathematics modules are as follows:

Upper Primary Science
Module 1: *My Built Environment*
Module 2: *Materials in my Environment*
Module 3: *My Health*
Module 4: *My Natural Environment*

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Module 2: *Energy Use in Electronic Communication*
Module 3: *Living Organisms' Environment and Resources*
Module 4: *Scientific Processes*

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Module 1: *Teaching Technology in the Primary School*
Module 2: *Making Things Move*
Module 3: *Structures*
Module 4: *Materials*
Module 5: *Processing*

Junior Secondary Technology
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Module 2: *Systems and Controls*
Module 3: *Tools and Materials*
Module 4: *Structures*

Upper Primary Mathematics
Module 1: *Number and Numeration*
Module 2: *Fractions*
Module 3: *Measures*
Module 4: *Social Arithmetic*
Module 5: *Geometry*

Junior Secondary Mathematics
Module 1: *Number Systems*
Module 2: *Number Operations*
Module 3: *Shapes and Sizes*
Module 4: *Algebraic Processes*
Module 5: *Solving Equations*
Module 6: *Data Handling*

A MESSAGE FROM THE COMMONWEALTH OF LEARNING

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Dato' Professor Gajaraj Dhanarajan
President and Chief Executive Officer

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Introduction

Welcome to *Solving Equations*, Module 5 of Teaching Junior Secondary Mathematics! This series of six modules is designed to help you to strengthen your knowledge of mathematics topics and to acquire more instructional strategies for teaching mathematics in the classroom.

The guiding principles of these modules are to help make the connection between theoretical maths and the use of the maths; to apply instructional theory to practice in the classroom situation; and to support you, as you in turn help your students to apply mathematics theory to practical classroom work.

Programme Goals

This programme is designed to help you to:

- strengthen your understanding of mathematics topics
- expand the range of instructional strategies that you can use in the mathematics classroom

Programme Objectives

By the time you have completed this programme, you should be able to:

- develop and present lessons on the nature of the mathematics process, with an emphasis on where each type of mathematics is used outside of the classroom
- guide students as they work in teams on practical projects in mathematics, and help them to work effectively as a member of a group
- use questioning and explanation strategies to help students learn new concepts and to support students in their problem solving activities
- guide students in the use of investigative strategies on particular projects, and thus to show them how mathematical tools are used
- guide students as they prepare their portfolios about their project activities

How to work on this programme

If you have reached this point in the course while faithfully doing the Assignments and sometimes trying out the new concepts in your classroom… congratulations! That personal effort will have already enriched your classroom work, benefitting your students now and for years to come.

Continue to "do the math" like a good student as you approach the end of the six-module series, and interact with teaching colleagues to gain from their insights. For example, this module makes use of a domino game to teach algebra, as Module 2 did when teaching number operations. What do your colleagues think of using dominoes (or other games) in a Maths class?

ICONS

Throughout each module, you will find the following icons or symbols that alert you to a change in activity within the module.

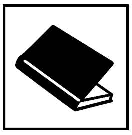
Read the following explanations to discover what each icon prompts you to do.

	**Introduction**	Rationale or overview for this part of the course.
	**Learning Objectives**	What you should be able to do after completing this module or unit.
	**Text or Reading Material**	Course content for you to study.
	**Important—Take Note!**	Something to study carefully.
	**Self-Marking Exercise**	An exercise to demonstrate your own grasp of the content.
	**Individual Activity**	An exercise or project for you to try by yourself and demonstrate your own grasp of the content.
	**Classroom Activity**	An exercise or project for you to do with or assign to your students.
	**Reflection**	A question or project for yourself—for deeper understanding of this concept, or of your use of it when teaching.
	**Summary**	
	**Unit or Module Assignment**	Exercise to assess your understanding of all the unit or module topics.
	**Suggested Answers to Activities**	
	**Time**	Suggested hours to allow for completing a unit or any learning task.
	**Glossary**	Definitions of terms used in this module.

CONTENTS

Module 5: Solving equations

Module 5
Solving equations



Introduction to the module

Algebra can be a useful tool to describe and model real life situations. Models are always a simplified form of reality but do help to make sense of our environment. The models might be linear, quadratic, exponential or other and frequently lead to solving of equations. If, for example, the length L cm of a spring stretched by a mass M g is found from experiments to be modelled by $L = 12 + 0.1M$, the question can be asked: what mass attached to the spring will double its length? The equation $24 = 12 + 0.1M$ is to be solved to answer the question.

Aim of the module

The module aims at:

a) increasing your knowledge of the basic concepts of equations
b) practising pupil centred teaching methods in the learning of solving of equations
c) raising your appreciation for use of games and investigations in pupils' learning of mathematics
d) reflection on your present practice in the teaching of solving of equations
e) making you aware of a variety of learning aids that can be used in the teaching of solving equations
f) aquainting you with different techniques that can be used in solving of equations
g) introducing you to the first steps in the solution of the cubic equation

Structure of the module

This module first treats the basic concept of 'equation' and the core problems pupils encounter in learning about equations and solving them (Unit 1). Unit 2 covers a wide variety of techniques to solve linear equations and suggests activities to be used in the classroom for the learning and consolidation of solving linear equations. Unit 3 covers the techniques for solving quadratic equations and Unit 4 looks at simultaneous linear equations. The emphasis is on looking at activities for the classroom to make solving equations more meaningful and understandable for pupils. A pupil centred approach is encouraged throughout the module by activities that can be set to pupils and will actively involve them in constructing mathematical knowledge and applying mathematics in a variety of situations. Unit 5 covers the (partial) solving of cubic equations as an extension of content knowledge—considerably beyond topics of Junior Secondary Mathematics.

The module does not yet make use of the current available technology: graphic calculators and symbolic-manipulation algebraic calculators. This new technology is presently changing dramatically the study of algebra in the western world: from symbolic paper-and-pencil manipulation towards conceptual understanding, symbol sense and mathematical modelling of real life problems.

Objectives of the module

When you have completed this module you should be able to create with confidence, due to enhanced background knowledge, a learning environment for your pupils in which they can:

(i) acquire various techniques to solve equations

(ii) consolidate through games the solving of simple linear equations

(iii) use algebra to model realistic situations

Prerequisite module

This module assumes you have covered Module 1, especially the unit on sequences and finding the general rule for the *n*th term in a sequence and Module 4, the basic manipulative techniques in algebra.

Unit 1: Equations

Introduction to Unit 1

In this unit you will learn about the basic concepts involved in solving equations. Before moving to techniques to solve equations you have to be aware of what equations are and why they require attention at the lower secondary level. Pupils do have difficulties in understanding basic concepts of equations mainly because such concepts are not adequately covered in the lessons.

Purpose of Unit 1

The main aim of this unit is to look at the basic questions: What are equations? How are they different from expressions and identities? What problems are encountered in the learning of solving of equations?

Objectives

When you have completed this unit you should be able to:

- distinguish between statements and expressions, illustrating with examples and non examples
- distinguish between equations and identities, illustrating with examples and non examples
- distinguish between equal symbol, identity symbol and equivalent symbol
- set activities for pupils to enhance understanding of the equal sign

Time

To study this unit will take you about six hours.

Unit 1: Equations

Section A1: Teaching and learning equations

The emphasis in algebra is frequently on techniques: techniques to manipulate expressions, techniques to solve equations. What remains obscure to many pupils is the purpose of these activities (other than for examination purposes: examination oriented learning / teaching). In the first part of this course (Module 1) you looked at what algebraic knowledge might be of use to pupils. This resulted in the following characteristics for the teaching of algebra.

The teaching and learning of algebra should be:

(i) applicable to situations pupils are likely to meet in the world of work (for example substituting values in a given (word) formula)

(ii) emphasising the interpretation of situations expressed in algebraic format (obtaining information from graphs, making sense of word formulas)

(iii) geared towards general techniques rather than case specific techniques (trial and improvement method of solving equations to be preferred over, among others, the quadratic formula with its limited application)

(iv) related to the use of algebra to describe patterns and relationships pupils are interested in

(v) introduced when there is a felt need for it (formal techniques to solve equations should be introduced when other informal techniques fail)

Throughout this section, when looking at the teaching and learning of equations the above points should be kept in mind.

Section A2: Basic concepts: equation, variable, equal symbol

When dealing with solving of equations a number of concepts have to be clearly understood. The following concepts have to be covered:

(i) what is a variable? (this was discussed in Module 1)

(ii) what is an equation?

(iii) meaning of the = or equality symbol

(iv) meaning of the ⇔ or equivalence symbol.

At this level you might feel there is no need to introduce the equivalent sign to pupils.

What is an equation?

To get clear what is meant by an equation first we look at statements.

Statements

A statement is an expression (in words, numerals or algebraic) which might be true or false. Any expression of which you can ask is it true or false (although you are not necessarily able to tell whether the statement is true or false) is a statement.

Examples of statements:

1. I am 62 years old.
2. More boys than girls are born.
3. $7 > 5$
4. $2 + 5 = 9$
5. A backward animal is called a lamina.
6. Mathematicians do it with numbers.
7. Even the biggest triangle has only three sides.
8. A parallelogram is a trapezium.
9. $2x + 4 = 7$
10. She is the engineer in town.
11. $8 \leq 8$
12. More females are HIV positive than males.
13. The area of the farm is 40 ha.
14. She ran the 100 m in 6.75 seconds.
15. Girls perform better in mathematics in form 1 secondary school than boys.
16. Politicians cater more to themselves than for the people in the country.
17. Jesus was not born on Christmas day.
18. $2 + 2 = 2 \times 2$
19. $3 + 3 = 3 \times 3$
20. $a + a = a \times a$

All the above are statements because you can sensibly pose the question "Is it true?" Some are true, others false and some are sometimes true sometimes false.

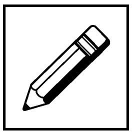
Non examples:

What is your name?

Go and multiply!

Well done!

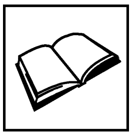
Have a good time.



Self mark exercise 1

1. Which of the above statements 1 - 20 are true? Which are false? Which are sometimes true and sometimes false?
2. Write down three examples of statements that are (i) always true (ii) always false (iii) sometimes true.
3. Write down three examples of non statements.

Check your answers at the end of this unit.



True, false and open statements.

A statement can be **true**.

e.g., Human beings can never become 3.5m tall.

A statement can be **false**.

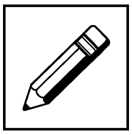
e.g., The moon's surface is made of butter.

A statement can be **open**.

We are not able to say whether it is true or false. It might be true in some situations or false in others

e.g., The maximum temperature was 32˚C

$x + y = 8$



Self mark exercise 2

Which of the above statements 1 - 20 are open?

Check your answers at the end of this unit.

Equations

Equations are open statements of equality. You could also say that an equation is two expressions linked with an equal sign, involving at least one variable.

For example

$3x - 6.7 = {}^{-}4.5$ $\quad$ $x + 3y = 3x - 2y + 7$ $\quad$ $\sqrt{x} = 2x + 7$

$3a + 2b$ is NOT an equation but an (algebraic) expression. It does not contain an equality sign.

$3 + 4 = 8$ and $4 + 4 = 8$ are NOT equations as there is no variable involved in the statement.

Solving equations

Solving an equation means trying to find all the possible values of the variable(s) that make the statement true. It is generally assumed that equations are solved over R, the set of real numbers. At times, however, you might be solving over a different set, as a decimal or fractional or negative values might not make sense in the given context.

For example, suppose you are trying to find the height x of an open box you can make from a given piece of cardboard paper which will have a maximum capacity. If this would lead to the quadratic equation $x^2 - 2x - 80 = 0$, this factorises as

$(x + 8)(x - 10) = 0$, leading to $x = 10$ and $x = {}^{-}8$. In this case the solution $x = {}^{-}8$ would not make sense as the height of a box cannot be negative.

Some equations are true for none, one or more value(s) of the variable(s) but not for ALL values of the variable(s). This type of equations are called conditional equations.

The value(s) that make an equation to become true are called the roots or the solutions of the equation.

Here are some examples of conditional equations:

1. $2x - 4 = x + 2$
2. $x^2 + 4x = 0$
3. $2x + 9 = 3.75$
4. $x + y = 4$
5. $2x = y$
6. $x^2 = 9$
7. $\sqrt{x} = 2.5$
8. $x^2 + y^2 = 25$
9. $x^2 + 1 = 0$

Self mark exercise 3

1. How many solutions has each of the above equations when solving over the set of (i) natural numbers (ii) integers (iii) rational numbers (iv) real numbers?
2. For each equation state one value for the variable(s) that do(es) not satisfy the equation.

Check your answers at the end of this unit.

Identities

An equation that is true for ALL values of the variable(s) is an identity.

For example:

$5a + 2a = 7a$ is an identity, because it is true whatever value you take for a.

Also $(p + q)^2 = p^2 + 2pq + q^2$ is an identity.

Some authors use ≡ in identities, in order to clearly distinguish between a conditional equation and an identity. They write, for example:
$(a^2 - b^2) \equiv (a + b)(a - b)$

Self mark exercise 4

Which of the following algebraic statements are true, false, open (conditional) or an identity?

1. $5x - 10xy = 5(x - 2y)$
2. $27 \le 3^3$
3. ${}^-4 > {}^-3$
4. $(a + b)^2 = a^2 + b^2$
5. $\sqrt{9} = 3$
6. $3a + 2b = 5ab$
7. $x^2 + 1 > 0$
8. $8a^4 \div 4a^3 = 2a$
9. $x + y = 49$
10. $x - x = 0$
11. $\sqrt{a^2} = a$
12. $\sqrt{a^2 + b^2} = a + b$

Check your answers at the end of this unit.

The Equality sign (=)

Many teachers take the use (misuse?) of the symbol = for granted and do not spend time on consolidation of the concept of equality. They feel that there is no problem here. However research has indicated otherwise: many pupils do not have the correct concept as to the meaning of the equality symbol.

The basic concept required is that the equal sign is **symmetric and transitive**.

Symmetric means that you can read from both sides:
$2 + 2 = 4$ but also $4 = 2 + 2$.

Transitive means that if $a = b$ and $b = c$ then also $a = c$.

The common misinterpretation of the equal sign is as a “do something” sign—an instruction to place behind the symbol “the answer”.

Pupils are reluctant to accept statements such as

$7 = 5 + 2$ (the answer should be on the right it is felt)

$4 + 3 = 6 + 1$ (pupils feel “the answer” 7 is to be added)

In algebra, pupils might feel that expressions such as $a + b$ cannot be ‘the answer’ as they argue that you still have to add. This leads to the conjoining error discussed in section 3.1.4.

The equal sign is generally poorly understood and requires special attention and reinforcement during the learning of mathematics in general and of algebra specifically.

An activity with pupils to make them understand that the equal sign is not an instruction to do something but expressing that two expressions have the same value is the following:

Pupils can be asked to create arithmetic equalities (identities in fact)

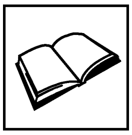
(i) using a single operation at both sides of the equal sign, make as many equations as you can with the value 12.

e.g., $2 \times 6 = 4 \times 3$ $\quad$ $2 \times 6 = 10 + 2$

(ii) using several operations at both sides of the equal sign, make as many equations as you can with the value 12.

e.g., $7 \times 2 + 3 - 5 = 5 \times 2 - 1 + 3$

$4 \times 12 \div 24 \times 6 = 6 \times 16 \div 24 \times 3$



Equivalent sign

The equivalent symbol, $\Leftrightarrow$ is used to express that two statements (for example, two equations) are identical. In terms of equations the equivalent sign is expressing that two equations have the same solution.

For example $4x - 3 = x + 6$ and $3x = 9$ are equivalent equations.

So are $4x - 3 = x + 6$ and $x + 24 = 9x$ as both have the same solution.

You can write this as $4x - 3 = x + 6 \Leftrightarrow 3x = 9$

and $4x - 3 = x + 6 \Leftrightarrow x + 24 = 9x$ respectively.

Equivalent sign relate equations to each other.

A common error among pupils is confusing the equivalent sign ($\Leftrightarrow$) and the equal sign. They misuse the equal sign, using it in places where the equivalent sign is to be used.

A pupil might wrongly write:

$$5x + 7 = 2x - 5$$

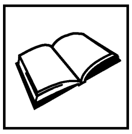
$$= \quad 5x + 7 - 2x = 2x - 5 - 2x$$

$$= \quad 3x + 7 = {}^{-}5$$

$=$ ...... etc.

To reinforce the concept of equivalence pupils could be asked to create as many equations as they can all with the solution $x = 3$. The alternative at this age is to ignore the equivalence sign altogether.

Section A3: Equations in context

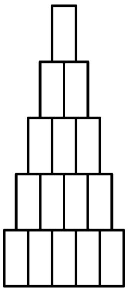


Equations arise in context / real life situations of generalised arithmetic if the question is posed in reverse. These situations should be looked at first, i.e., forming equations comes before the solving of the equations. Once pupils discover that equations are the result of modelling / describing realistic situations, the next step will be: How can these equations be solved?

This means that 'word problems' is not a separate section, an application of techniques to solve equations covered in isolation. On the contrary 'word problems' are the starting point and in order to answer these questions resulting from 'real' situations, techniques are required. The need to develop and learn a technique is prior to the technique to be used. First the need to solve a quadratic equation is to be created—pupils must feel a need to learn a technique—before looking at different techniques that can be used to solve quadratic equations.

For example:

Pupils have noted that in supermarkets tins are at times displayed in stacks as illustrated.



They wonder how many tins there are altogether in 10 layers or more in general in a stack of n layers.

If a general expression has been obtained for the number of tins in the triangular stack and it has been found that the total number of tins in a triangular stack is given by

$$\frac{1}{2}(\text{number of layers})(\text{number of layers} + 1),$$

the question might arise: can I place 276 tins in a triangular stack? How many tins do I have to place on my bottom layer?

This will present pupils with the quadratic equation $\frac{1}{2}n(n + 1) = 276$ and the question: How can we solve this equation?

The techniques of solving equations must serve a purpose: trying to answer realistic questions. Studying the techniques is secondary; primary is modelling situations into relationships. Word problems are not to be an appendix to the solving of equations but the starting point. Having expressed a situation in equation form, pupils will be motivated to solve these equations and ready for techniques to do so. Initially these techniques might be very simple: trial and error method (pupils "see" the answer immediately and only need to check the answer they "saw"). To create a "need" for techniques to solve equations the examples should be selected with care. Most 'starting questions' in many books do not really need equations at all. Pupils will argue: why do a complicated thing if you can 'see' it immediately?

For example:

a) I bought a book for P22.00 and later met a friend who I gave back P5.00 I borrowed from him some time ago. Now I had P 8.50 in my purse. How much did I have before entering the book shop?

Here there is NO NEED for an equation. Pupils will be reasoning backwards:

P8.50 + P5.00 = P13.50

P13.50 + P22.00 = P35.50

So I had P35.50 in my purse.

The presentation given by pupils frequently violates the symmetry and transitivity of the equal sign, when they write:
8.50 + 5 = 13.50 + 22 = 35.50. ('stringing')

Self mark exercise 5

1. a) Solve the tin stack problem.

   Derive the suggested formula for the number of tins in an n layer stack.

   b) Find the number of layers and the number of tins in the bottom layer if 276 tins are in the stack.

2. Describe in detail the remedial steps you would take in order to help a pupil presenting its working as shown above in example a. to overcome the ‘stringing’ error presentation.

Check your answers at the end of this unit.

b) A video shop offers customers videos on two rental plans. In the first plan you pay a subscription fee of P32.50 per year and for every video hired you pay P5.00. The second plan has no subscription fee but you pay P7.50 per video hired.

For what number of rental videos will the first plan be cheaper?

Pupils might try with trial and error, but fail to find the number of videos at the ‘break even point’. In equation format, taking the number of videos hired for which both rental plans give the same cost as n, you will get $32.50 + 5n = 7.5n$

This might be one example in which pupils feel the need to learn a technique to solve an equation of the format $ax \pm b = cx \pm d$.

Presenting a good number of situations leading to equations of the format $ax \pm b = cx \pm d$ will motivate pupils to learn several techniques to solve such type of equations.

c) Here is another situation leading to a linear equation. Without forming an equation and a technique to solve the equation formed, the problem is ‘hard’—you cannot just ‘see’ the answer.

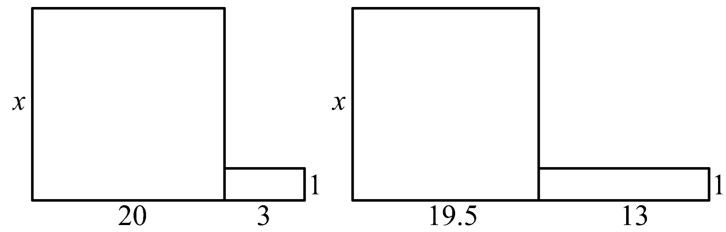
The point of no return.

Imagine that you are the pilot of a light aircraft which is capable of cruising at a steady speed of 300 km/h in still air. You have enough fuel on board to last you four hours. You take of from an airfield and, on the outward journey, are helped along by a 50 km/h wind increasing your cruising speed relative to the ground to 350 km/h. Suddenly you realise that on your return journey you will be flying into the wind and will therefore only be able to fly at a ground speed of 250 km/h.

- what is the maximum distance you can travel from the airfield and still be sure you have enough fuel left to make a safe return journey?
- when will you need to turn around?

d) As a last example of linear equations in context we look at: using area of plots as a context for linear equations.

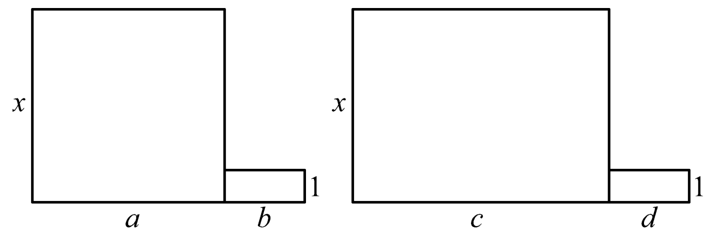
Below are illustrations of plots owned by two persons, with the measurements in metres. The areas of their plots are the same. What is the depth x m of their plot?



Area: $20x + 3$ Area: $19.5x + 13$

The equation to be solved is in this case $20x + 3 = 19.5x + 13$.

The general diagram for the equation $ax + b = cx + d$ is illustrated below.



Area: $ax + b$ Area: $cx + d$

Unit 1, Practice activity

1. Solve the "point of no return" question.
2. Investigate the 'area model' to represent linear equations. Using illustrations show which type of linear equations can be represented with the model. What are the limitations of the model, i.e., in which cases does it fail?
3. Develop a worksheet with 10 realistic situations, as in the examples above, leading to linear equations in order to create a need for pupils to learn techniques for solving linear equations of the format $ax \pm b = cx \pm d$.
4. Try out in your class the worksheet you designed.
5. Write an evaluative report.

Present your assignment to your supervisor or study group for discussion.

Summary

This unit has suggested ways to teach what is an equation and what is not. It stressed that both examples and counterexamples will help your students to understand "equals", and that the examples should be both "real and complex": realistic and concrete enough to make sense, but also complex enough that the student has to grapple with the equals concept in order to solve them.

Unit 1: Answers to Self mark exercises

Self mark exercise 1

1. - depends who makes the statements -
2. T
3. T
4. F
5. - depends on interpretation - T
6. S
7. t
8. - depends on definitions - T (or F)
9. S
10. S
11. T
12. S
13. S
14. F
15. nearly always T
16. S
17. F
18. T
19. F
20. S

Self mark exercise 2

Open are 1, 6, 9, 10, 12, 13, 15, 16, 20

Self mark exercise 3

1.

Equation	1	2	3	4	5	6	7	8	9
No. of sol. in N	1	0	0	3	∞	1	0	2	0
No. of sol. in Z	1	2	0	5	∞	2	0	4	0
No. of sol. in Q	1	2	1	∞	∞	2	1	∞	0
No. of sol. in R	1	2	1	∞	∞	2	1	∞	0

2. Not true for $x = 0$ $x = 1$ $x = 0$ (0, 0) (1, 1) x=2 $x = 1$ (0, 0) $x = 1$

Self mark exercise 4

1. identity
2. true
3. false
4. open
5. true
6. open
7. identity
8. open
9. open
10. identity
11. open
12. open

Self mark exercise 5

1. a)

Number of layers	1	2	3	4	5	6	...	n
Number of tins	1	3	6	10	15	21	...	$\frac{1}{2}n(n+1)$

Use the method of differences to find the general term of the sequence of triangular numbers.

b) 24 layers, bottom layer 24 tins

2. (i) Ask pupil to explain the working (diagnosing the error).

(ii) Ask pupil to compare 8.50 + 5 and 35.50 which are ‘equal’ by pupil’s statement (creating conflict).

(iii) In discussion with pupil form the correct concept, i.e., writing separate steps.

(iv) Set similar examples to consolidate correct presentation of working.

Unit 2: Solving linear equations

Introduction to Unit 2

For many ages, the focus of algebra has been on solving equations. Techniques were developed to solve various types of equations. Some of the techniques developed over time are part of the lower secondary mathematics curriculum. Although fluency in solving linear and quadratic equations might be an objective, techniques should not be followed blindly. A structural approach needs to be followed. Many pupils solve without problems an equation such as $4x + 7 = 35$. If asked to find the value of $2t + 1$ if $4(2t + 1) + 7 = 35$, few recognise the structure and solve directly for $2t + 1$. Most solve for t and then find the value for $2t + 1$.

Purpose of Unit 2

The main aim of this unit is to look at techniques of solving equations that follow a structural approach. The work in module 4 (simplifying, expanding and factorising algebraic expressions) are tools that will be used when appropriate in the solving of equations. The starting point is modelling real life situations leading to equations, i.e., creating a need to solve equations. In this unit you will look at linear equations and techniques to solve them, comparing the advantages and limitations of each method. The emphasis will be: how can you assist pupils in acquiring relational knowledge about these concepts, and use games to consolidate solving of linear equations.

Objectives

When you have completed this unit you should be able to:

- solve linear equations by the following methods:
  - cover up
  - reverse flow
  - balance
  - trial and improvement
  - graphical
  - using algebra tiles
- demonstrate awareness of the advantages and limitations of the above listed methods
- set questions to pupils to create the need to learn techniques to solve equations
- make and use domino sets to consolidate solution of linear equations
- use a puzzle context for forming and solving equations
- set activities to your class for pupils to consolidate solving of linear equations
- use teaching techniques to assist pupils to overcome problems in the learning of solving of equations

Time

To study this unit will take you about 15 hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topic.

Unit 2: Solving linear equations

Section A1: Cover-up technique

The construction of arithmetic identities naturally moves first into a solving technique for equations by using “blot” games: an arithmetic identity with a hidden number.

In identities such as:

$7 \times 2 + 4 = 5 \times 3 + 3$ $\qquad$ $7 \times 3 + 7 = 5 \times 6 - 2$

the teacher might cover with her hand(s) the 7s.

Next the covered/hidden number might be replaced by a ‘blot’ / box / triangle, etc.

Finally the missing number can be represented by a letter variable.

Example of **“blot” game** for pupils:

You had done your assignment but your 3 year old brother played with the whiteout fluid and blotted away numbers. Can you find them again?

The numbers blotted away are all integers. If in one problem more than one blot appears, under the same type of blot you find the same number. Different type of blots might cover the same or different numbers.

Discussions on how pupils found the ‘missing’ numbers, the number of possible answers, will be first steps in solving techniques of equations.

a. $258 + \bigcirc = 280$
b. $\bigcirc - 43 = 300$
c. $100 - \bigcirc = 89$
d. $995 = 199 \times \bigcirc$
e. $\bigcirc \times 7 = 784$
f. $\bigcirc + \bigcirc = 12$
g. $\square \times \bigcirc = 49$
h. $\dfrac{\bigcirc}{\bigcirc} = 1$

The cover-up technique is a structural approach. Pupils are to understand the structure of the equation. Research has indicated that pupils knowing this technique out perform pupils that use the transpose model for solving equations. The cover-up technique requires relational/structural understanding; the transpose technique is frequently ill understood and ‘blindly’ applied as a memorised rule (operational understanding).

Mathematics in general is describing relationships and patterns. It is therefore important that pupils learn to recognise the same structure in situations that might be dissimilar at a first glance.

Pupils are to learn to look for structure. The manipulative and rule oriented approach discourages such a view.

For example, if pupils meet equations such as:

(i) $3x - 18 = 0$ (ii) $3(t + 2) - 18 = 0$ (iii) $3(a + 3)^2 - 18 = 0$

few pupils notice that the structure is identical and hence that ALL can be solved in the way one would solve equation (i). Most pupils solve (ii) & (iii) by removing brackets first etc. Instead of deducing from (i) that if $x = 6$, then in (ii) $t + 2 = 6$, hence $t = 4$ and in (iii) $(a + 2)^2 = 6$, hence

$a + 2 = \sqrt{6}$ or $a + 2 = {}^{-}\sqrt{6}$, leading to the exact values of a.

The last equation is seen as $3\,\square - 18 = 0$, the box covering the $(a + 3)^2$. In the box what is needed is 6, hence $(a + 2)^2 = 6$.

Here is another example: solving $5(p + 3) - 6 = 19$, using the cover-up method.

$5(p + 3) - 6 = 19$

$\bigcirc - 6 = 19$

You must have covered 25

So $5\ (p + 3) = 25$

$5\ \bigcirc = 25$

Covered: 5

So $p + 3 = 5$

$\bigcirc + 3 = 5$

Covered 2

So $p = 2$

Similarly solving the non linear equation $69 - \dfrac{96}{7 - \Delta} = 37$.

Cover $\dfrac{96}{7 - \Delta}$ so it reads $69 - \bigcirc = 37$.

Covered must be "32".

So we now know that $\dfrac{96}{7 - \Delta} = 32$.

Cover the denominator $7 - \Delta$. So it reads $\dfrac{96}{\bigcirc}$.

You must have covered 3.

So $7 - \Delta = 3$.

Which gives as a result $\Delta = 4$.

Such a ‘structured’ based approach is of more benefit than an operational approach.

The last example illustrates that the cover-up method is not restricted to linear equations. The restriction of the method is that the variable is to appear in ONE place in the equations only. Trying to solve $x^2 - x = 0$ by cover-up method will fail.

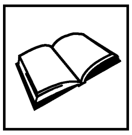
Self mark exercise 1

Using the cover up method solve the following equations:

1. $3(x - 4) - 6 = 12$
2. $\dfrac{3}{x-4} - 6 = 123$
3. $(x - 4)^2 = 12$
4. $\dfrac{4}{6-x} - 3 = 5$
5. $x^3 - 7 = 118$
6. $\dfrac{3x-1}{5} - 8 = 5$
7. $12 - x^2 = 4$
8. $(12 - x)^2 = 4$

Check your answers at the end of this unit.

Section A2:
"Undoing"—working backwards using flow diagrams



"Guess my number" game and "I will tell you the answer you have got" games lead to describing the operations in a flow chart and reversing the flow to find the answer.

"Guess my number" example:

I am thinking of a number. If I double my number and subtract 6, I get 9. What number am I thinking of?

See the illustration below. The first line illustrates 'what is said', the second line reverses the flow to answer the question.

$$\text{number} \xrightarrow{\times 2} 2 \times \text{number} \xrightarrow{-6} 2 \times \text{number} - 6 \xrightarrow{\qquad} 9$$

$$7.5 \xleftarrow{\div 2} 15 \xleftarrow{+6} 9$$

"I will tell you the answer you have got" example:

Think of a number, multiply your number by 3, subtract 6. What answer did you get? (Pupil answers: for example 9) Then the number you were thinking of was 5.

See the illustration below:

$$\text{number} \xrightarrow{\times 3} 2 \times \text{number} \xrightarrow{-6} 2 \times \text{number} - 6 \xrightarrow{\qquad} 9$$

$$5 \xleftarrow{\div 3} 15 \xleftarrow{+6} 9$$

Here is another example to solve $4(x - 2) + 3 = 5$.

In words: I think of a number, subtract 2, multiply the answer by 4 and add 3. The answer is 5.

In a flow chart:

$$x \xrightarrow{-2} x - 2 \xrightarrow{\times 4} 4(x - 2) \xrightarrow{+3} 4(x - 2) + 3 = 5$$

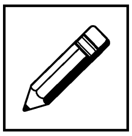
To find the number, reverse the flow, starting at 5:

subtract 3 (gives 2), divide by 4 (gives $\frac{1}{2}$) and add 2 (gives $2\frac{1}{2}$).

$$2\frac{1}{2} \xleftarrow{+2} \frac{1}{2} \xleftarrow{\div 4} 2 \xleftarrow{-3} 5$$

The flow chart method model requires understanding of the structure of the relationship as a chain of operations and knowledge of the inverse of each operation. The examples should be such—i.e. difficult enough—that pupils feel the need for a new technique. There is no point in asking pupils (at this level): I am thinking of a number and I add 2, now my answer is 36. What was my number?

Note that the method, as the previous one, is not restricted to linear equations. The method works on any equation with the variable in ONE place.



Self mark exercise 2

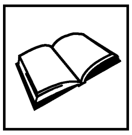
I. Using a flow chart model solve the following equations

1. $3(x-4)-6=12$
2. $\dfrac{3}{x-4}-6=12$
3. $3(x-4)^2=12$
4. $\dfrac{4}{6-x}-3=5$
5. $x^3-7=118$
6. $\dfrac{3x-1}{5}-8=5$
7. $12-x^2=4$
8. $(12-x)^2=4$

II. Why does the method not work with the variable in more than one place in the equation? Illustrate with examples.

Check your answers at the end of this unit.

Section A3: Trial and improvement technique



This is one of the most useful techniques in solving equations. As most equations cannot be solved exactly, approximation of the roots to a required degree of accuracy is more than appropriate in most practical situations.

Equations such as $\sin x = x$, $x^5 = x + 2$ cannot be solved exactly. Trial and improvement method will allow finding the roots to any degree of accuracy. The calculator, being available to all pupils these days, is a necessary tool.

This method starts by making a guess and substitutes values in the equation and checks whether the answer is above or below the required number.

As linear equations can be solved exactly, this method is not the most appropriate one for linear equations. Its power is with non linear equations.

For example:

a. $x^3 + 2x = 63$

Suppose you guess $x = 3$

Work the left hand side (LHS) and decide whether your guess is too small or too large for the required outcome of 63. Make a second guess to improve the outcome.

You could lay out your work as shown below.

Guess	LHS	RHS	too large or small?
3	33	63	small
4	72	63	large
3.8	62.47	63	small
3.85	64.77	63	large
3.82	63.38	63	large
3.810	62.93	63	small
3.812	63.017	63	large

The next guess could be 3.811 as 3.810 was too small and 3.812 too large, so you must search between those two.

b. A number together with 10 times the square root of the number is equal to 1000. What is the number?

In algebraic notation $n + 10\sqrt{n} = 1000$.

n	$n + 10\sqrt{n}$	
700	965	too small
800	1083	too big
750	1024	too big
725	994	too small
730	1000.2	too big
729	999	too small
729.5	999.6	too small
729.8	999.9	too small
729.9	1000.1	too big
729.85	1000.01	too big

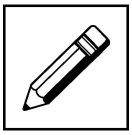
Depending on the accuracy one wants in the answer, the process can be continued. To 1 decimal place the number is 729.8.

This method is very general and works for all types of equations. Equations (the majority) that have no algebraic solution have to be solved by an numerical approximation method. It works neatly if you place the variables all at one side of the equal sign. For example to solve $x^3 + 6 = 2x^2 + 10$ it is easier to rewrite the equation as $x^3 - 2x^2 = 4$, and guesses for x are improved in each step to ‘hit’ 4 as closely as required, than to compare LHS ($x^3 + 6$) with RHS ($2x^2 + 10$) in each step.

The trial and improvement technique is very suitable when using the technology now generally available to all pupils.

a) graphic calculator (plotting the graphs of $y = n + 10\sqrt{n}$ and $y = 1000$ on the same axes system and zooming in on the point of intersection)

b) a spreadsheet using a search with ever finer increments



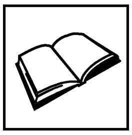
Self mark exercise 3

Using the trial and improvement technique, solve for x to 2 decimal places of accuracy, the following equations.

1. $x^3 - x^2 = 10$
2. $3\sin x - x = 0$ (Do not forget to place your calculator in Radian mode)
3. $\sqrt{2x} = 6 - x$

Check your answers at the end of this unit.

Section A4: Balance method model: performing the same operation on both sides, which could move into transposing techniques (“change side - change sign”)



This is the most commonly used algorithm for solving linear equations. The balance method model is a rather restricted algorithm. It works for linear equations only, unlike the methods discussed in the previous sections. The transposing technique is very much a manipulative technique based on ‘rules’ (change side - change sign) and far less a structural understanding. Many teachers present this technique as the only one, limiting the pupils by doing so. This technique (change side - change sign) has the tendency to become an ill understood operational technique, without clear understanding of the underlying structures.

Example 1

Solve for b the equation: $3(4 - 2b) - 4(b - 3) = 16$

$12 - 6b - 4b + 12 = 16$	expand
$24 - 10b = 16$	simplify, taking like terms together
$24 - 10b - 24 = 16 - 24$	subtract 24 from both sides (to get the term with the variable isolated)
$^-10b = -8$	simplify both sides
$\frac{^-10b}{^-10} = \frac{^-8}{^-10}$	divide both sides by $^-10$ (to get b)
$b = \frac{4}{5}$	

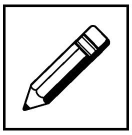
Example 2

$6(x - 1) - 5 = 4 - (2x - 3)$

$6x - 6 - 5 = 4 - 2x + 3$	expand LHS and RHS
$6x - 11 = 7 - 2x$	simplify
$2x + 6x - 11 = 7 - 2x + 2x$	add $2x$ to both sides
$8x - 11 = 7$	simplify (from here on civer-up / reverse flow methods would also work)
$8x - 11 + 11 = 7 + 11$	add 11 to both sides
$8x = 18$	simplify
$\frac{8x}{8} = \frac{18}{8}$	divide both sides by 8

$x = \frac{9}{4}$

$x = 2\frac{1}{4}$



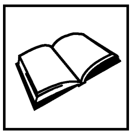
Self mark exercise 4

Solve, using the balance model:

1. $3(x - 4) = x + 3$
2. $2x - 3(x - 5) = 15 - x$
3. $\frac{2x - 4}{3} - \frac{x - 1}{2} = 1$
4. $2(4 - 3x) + 7 = 5(4 - 3x) - 5$
5. $15 - 3(2x - 1) = 2(4 - 3x) - 5$
6. $4(2x + 1) - 3 = 2(4x + 1) - 1$

Check your answers at the end of this unit.

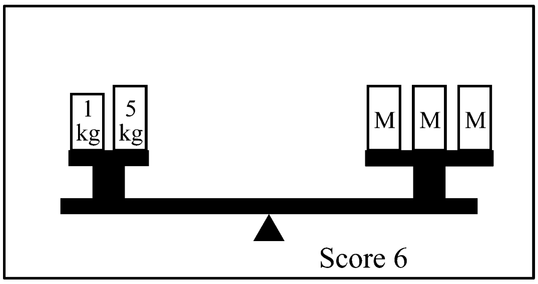
In the balance: consolidation game



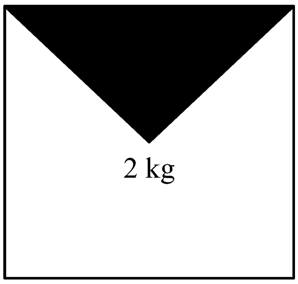
The balance technique can be practised and consolidated by using the "In the balance" game. The game at the same time enhances mental arithmetic.

You must make a set of Game Cards and a set of corresponding Mass Cards as illustrated below. (See following pages also.)

GAME CARDS



MASS CARD



Illustrated is the equation 3M = 6: three masses of M kg on the right hand scale and two masses one of 1 kg and the other 5 kg on the left hand scale. The winning mass card to be played is the 2 kg card as the solution to the equation is M = 2.

How to play the game:

Pupils in groups of 4 - 5 have a set of about 30 Game Cards. Each pupil has a set of mass cards. The Game Cards are upside down and one is opened. Each pupil tries to find the 'balancing' mass and place the mass card near the Game Card. The first pupil to play the correct mass card scores the indicated points.

If you make 6 different set of 30 cards a class can play for a long time. The difficulty level can be varied by allowing as mass cards (decimal) fractions or by placing the unknown masses at both sides of the balance. Game Cards can be made to illustrate any linear equation of the format $px + q = Px + Q$, with p, q, P and Q positive rational numbers. Ensure your value for x is always a positive rational. (You can also allow negative values, but that requires some story about a very special type of balance! Helium balloons have been suggested.)

Unit 2, Practice task

1. Make different sets of "In the Balance" game cards and corresponding mass cards.

   You need a set of 30 "In the Balance" cards for each group and one set of corresponding mass cards for each pupil.

   Make sets to consolidate solving equations of the format:

   Set 1: $ax = b$

   Set 2: $ax + b = c$

   Set 3: $ax + b = cx + d$

   Suggestions for each set are below this box. Blank balance card and mass card are on the next pages for photocopying.
2. Play "In the Balance" in your class. Write an evaluative report.

Present your assignment to your supervisor or study group for discussion.

Set 1:

Mass cards: 2kg, 2.5 kg, 3 kg, 3.5 kg, 4 kg, 4.5 kg, 5 kg, 5.5 kg and 6 kg (one set for each player)

Game cards (one set per group of players)

LH scale	RH scale	Score
M M M	2 kg, 7 kg	2
M M M	4 kg, 2 kg	2
5 kg, 2 kg	M M	3
M M M M	8 kg, 4 kg	2
6 kg, 9 kg	M M M	2
M M M	2 kg, 10 kg	2
M M	4 kg, 5 kg	3
2 kg, 3 kg	M M	3
9 kg, 9 kg	M M M	3
4 kg, 6 kg	M M M M	3
M M M M M	9 kg, 6 kg	4
M M M M	12 kg, 6 kg	4
M M M	10 kg, 5 kg	3
M M	4 kg, 7 kg	3
2 kg, 7 kg	M M	4
5 kg, 7 kg	M M M	2
3 kg, 11 kg	M M M M	4
16 kg, 14 kg	M M M M M	3
M M M M	14 kg, 8 kg	4
6 kg, 5 kg	M M	4

Set 2:

Mass cards: 2kg, 2.5 kg, 3 k,. 3.5 kg, 4 kg, 4.5 kg, 5 kg, 5.5 kg and 6 kg

Game cards

LH scale	RH scale	Score
M M M 5 kg	9 kg 8 k g	3
M M 3 kg	8 kg 2 kg	4
6kg 8 kg	M M M 5 kg	3
4 kg 6 kg	M M 6 kg	3
M M M M 5 kg	12 kg 7 kg	3
M M 9 kg	11 kg 7 kg	4
12 kg 9 kg	M M M 6 kg	4
12 kg 13 kg	M M M M 3 kg	4
M M 7 kg	12 kg 7 kg	4
M M M 10 kg	9 kg 7 kg	3
M M M M 6 kg 5 kg	11 kg	3
9 kg 6 kg	M M M M M 7 kg	3
7 kg 13 kg	M M M M 6 kg	4
9 kg 6 kg	M M M 3 kg	3
12 kg 6 kg	M M 9 kg	4
4 kg 16 kg	M M M 2 kg 3 kg	3
11 kg 5 kg 7 kg	M M 12 kg	4
M M M 6 kg	19 kg 5 kg	4
M M M M 7 kg	11 kg 14 kg	4
M M 8 kg	4 kg 5 kg 6 kg	4

Set 3:

Mass cards: 2kg, 2.5 kg, 3 k,. 3.5 kg, 4 kg, 4.5 kg, 5 kg,

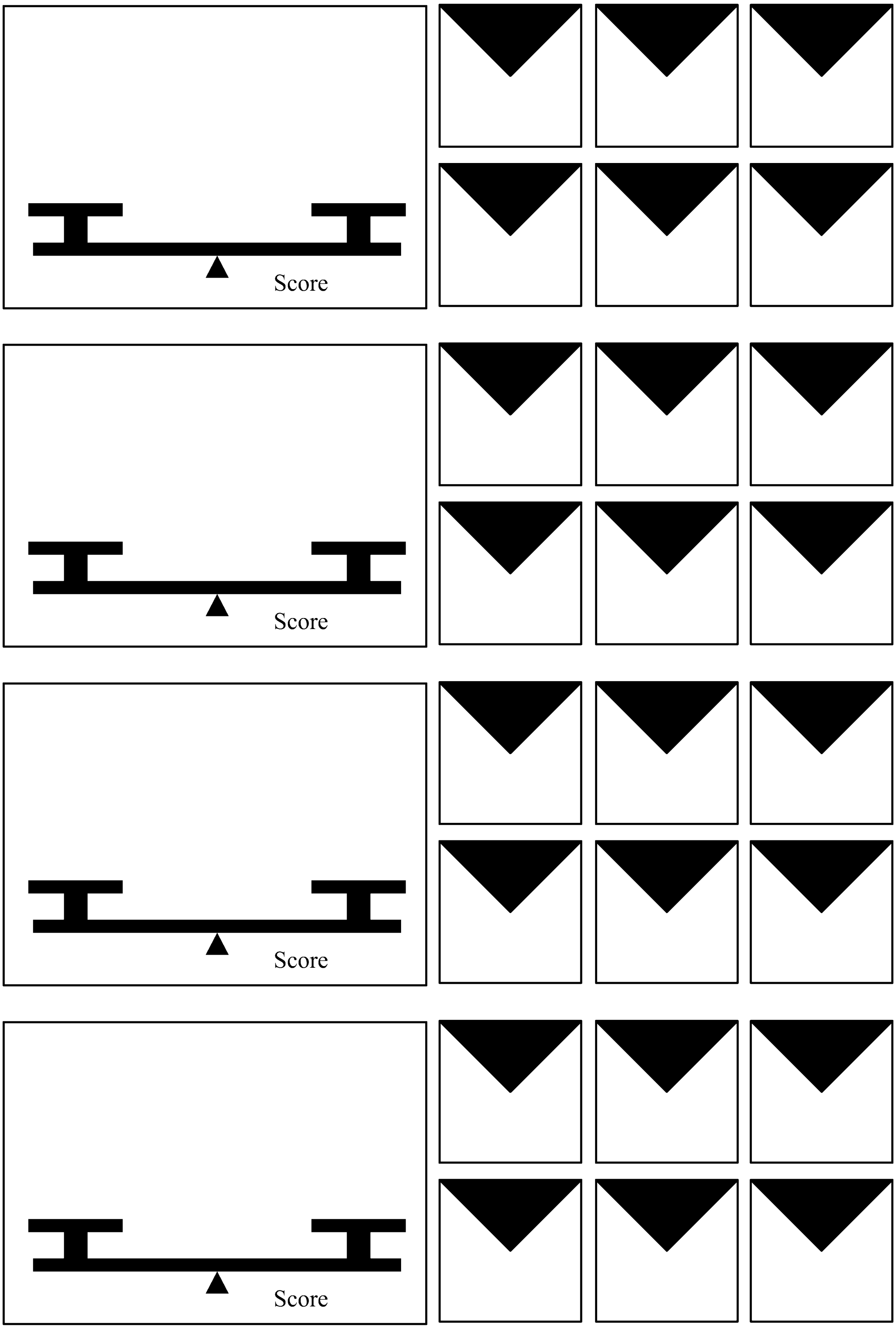
Game cards

LH scale	RH scale	Score
M M 4 kg	M M M 2kg	3
M M 9 kg	M M M M 4 kg	3
M 12 kg	M M 9 kg	3
M M M M 6 kg	M M 13 kg	4
M M M M 7 kg	M M 15 kg	4
M M M 4 kg	M 13kg	4
M M M 7 kg	M 17 kg	4
M M M M 3 kg	M M 11 kg	4
M 20 kg	M M M 8kg	4
M M 18 kg	M M M M 14 kg	3
M M M 8 kg	M 13 kg	3
M M M M 5 kg	M 14 kg	3
M M M M 9 kg	M M 16 kg	3
M M 11 kg	M M M 7 kg	4
M 9 kg 7 kg	M M M 7 kg	4
M M M 10 kg	M M M M 5 kg	4
M M 11 kg	M M M M 8 kg	4
M 12kg	M M M M 2 kg	4
M M M M 6 kg	M M 9 kg 8 kg	4
M M 7 kg 8 kg	M M M M 6 kg	4

Blank “In the Balance” card and mass card.

Balance card

Mass card

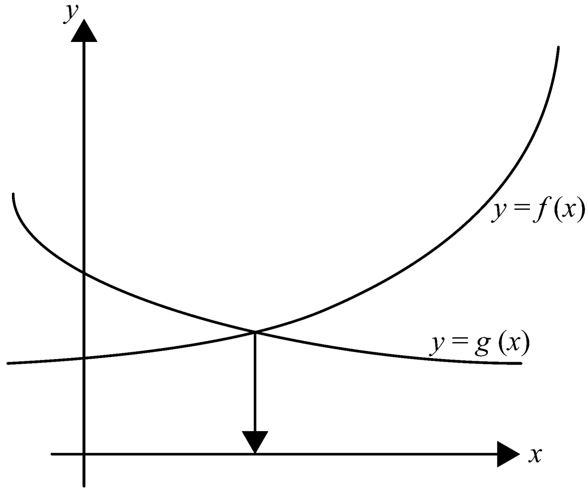


Section A5: Graphical techniques to solve equations

Using graphs to solve equations will become more important with the use of graphic calculators and computers. It also links graphs (of functions) to solving of equations. Solving an equation has as its graphical equivalent: finding the x-coordinate of the point of intersection of two graphs.

The method is applicable to all equations of a single variable.

If in general you have an equation $f(x) = g(x)$, with $f(x)$ and $g(x)$ expressions in x, then solving the equation is equivalent to finding the x-coordinate of the point of intersection of the graphs with equation $y = f(x)$ and $y = g(x)$.



At the point of intersection $f(x) = g(x)$, as the y-coordinates are the same. Hence the x-coordinate will give (one of) the root(s) of the equation.

The accuracy of the value for the root(s) will depend on the scale and accuracy of the graphs.

For example:

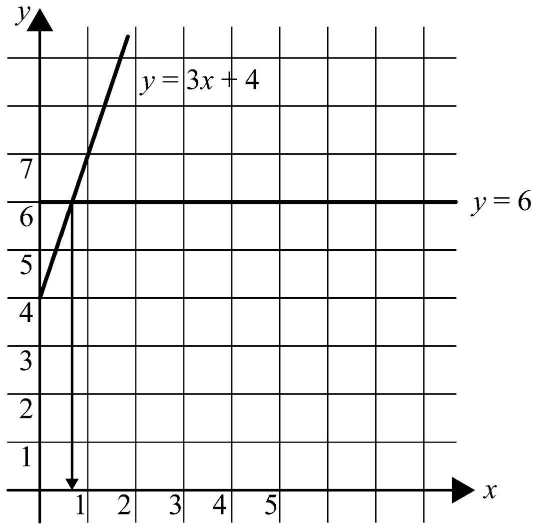
a) Solve $3x + 4 = 6$.

Plot the graph of $y = 3x + 4$ and the graph of $y = 6$.

Make a table with the coordinates of the points you are going to plot. For a linear graph two would be sufficient (why?), but to check on error calculations you always should take three.

x	0	1	2
$y = 3x + 4$	4	7	10

The x-coordinate of the point of intersection gives (an approximate) solution to the equation.



Using the graph you can read that $x = 0.7$ (1 decimal place)

b) $x^2 = x + 2$ can be solved by plotting the graphs of $y = x^2$ and $y = x + 2$ using the same pair of axes and looking for the x- coordinates of the point of intersection.

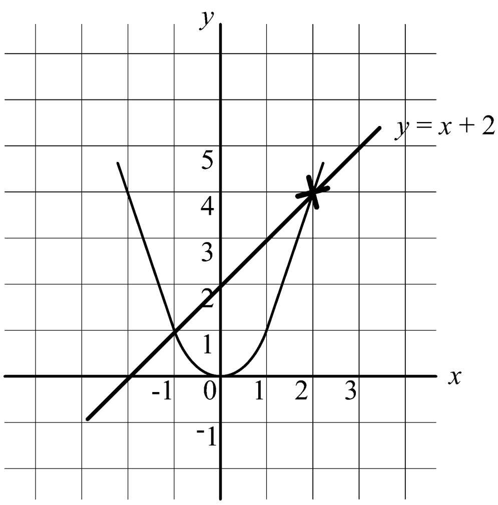
To plot a non-linear graph, take a good number of points (generally about 6 will do) and draw a smooth curve through the plotted points. The convention is to indicate a point as X. This is saying: the point is at the intersection of the two small lines.

To plot $y = x + 2$ use the table:

x	$^-2$	0	2
$y = x + 2$	0	2	4

To plot $y = x^2$ use the table:

x	$^-2$	$^-1.5$	$^-1$	0	1	2
$y = x^2$	4	2.25	1	0	1	4



Using the graph, the x-coordinates of the points of intersection are $x = {}^{-}1$ and $x = 2$.

Solution of the equation $x^2 = x + 2$ is therefore $x = {}^{-}1$ or $x = 2$.

Self mark exercise 5

1. The scale used in the above example was not very appropriate. (Why not?). Using graph paper and a more appropriate scale along the axes solve $3x + 4 = 6$ graphically. Try to read the answer to two decimal places.

2. Using graph paper and appropriate scales solve graphically $x^2 = x + 3$. The sketch in example b) above might guide you.

   Solve the following equations graphically.

3. $2x + 3 = 1 - 0.5x$ 4. $3\sin x - x = 0$ 5. $\sqrt{2x} = 6 - x$

Check your answers at the end of this unit.

Section A6: Using algebra tiles to solve linear equations

Algebra tiles can be used, especially to assist weaker pupils, to solve linear equations.

Needed: strips to represent x, ${}^{-}x$, ${}^{+}1$ and ${}^{-}1$. Photocopy the tiles on the following page and cut out.

Equations are represented by the strips.

Equations are physically solved by removing (or adding) equally valued pieces from (or to) each side of the picture.

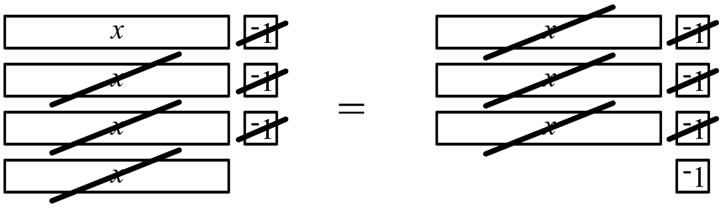
The basic idea is as with the negative integers: adding a strip representing x and one representing ${}^{-}x$ does not change the validity of the equation as it has zero value.

Example 1 : $4x + {}^{-}3 = 3x + {}^{-}4$

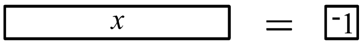
Step 1: representing the equation

$\boxed{x}$	$\boxed{{}^{-}1}$		$\boxed{x}$	$\boxed{{}^{-}1}$
$\boxed{x}$	$\boxed{{}^{-}1}$	=	$\boxed{x}$	$\boxed{{}^{-}1}$
$\boxed{x}$	$\boxed{{}^{-}1}$		$\boxed{x}$	$\boxed{{}^{-}1}$
$\boxed{x}$				$\boxed{{}^{-}1}$

Step 2: removing equally valued pieces from each side. The equally valued pieces to be removed have been crossed out.



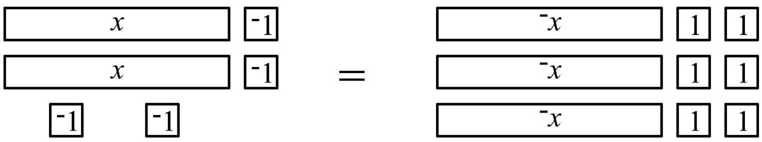
What remains after removing is:



The solution is therefore $x = {}^-1$

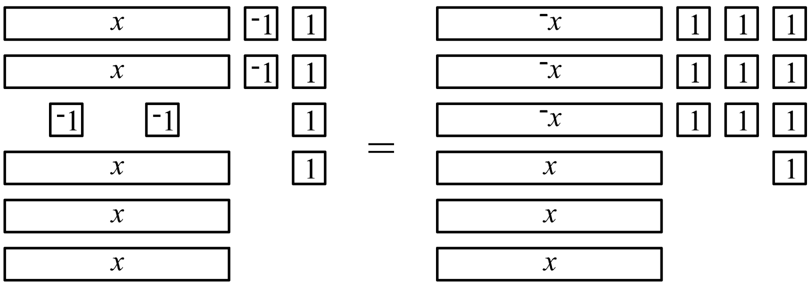
Example 2: solving $2x + {}^-4 = {}^-3x + 6$

Step 1: representing the equation

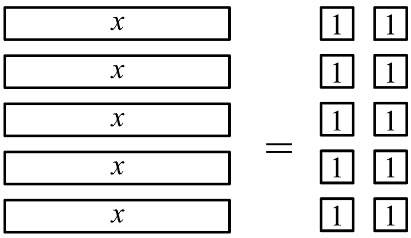


Step 2: adding equally valued pieces to each side

Add positive pieces to remove negative, i.e., 4 unit pieces and 3 x-pieces



What remains after removing pieces that cancel each other out is:



By arranging as in the diagram: $x = 2$

Self mark exercise 6

Use algebra tiles to solve these equations

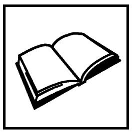
1. $4x + 3 = 5x - 2$
2. $6 - 2x = 3x - 4$
3. $5 - x = 8 - 2x$

Check your answers at the end of this unit.

Manipulatives for solving linear equations

x	1	^-x	$^-1$
x	1	^-x	$^-1$
x	1	^-x	$^-1$
x	1	^-x	$^-1$
x	1	^-x	$^-1$
x	1	^-x	$^-1$
x	1	^-x	$^-1$
x	1	^-x	$^-1$
x	1	^-x	$^-1$
x	1	^-x	$^-1$
x	1	^-x	$^-1$
x	1	^-x	$^-1$
x	1	^-x	$^-1$
x	1	^-x	$^-1$
x	1	^-x	$^-1$

Section B:
Games to consolidate the solving of linear equations



Games are a useful tool for consolidation of concepts and also for introduction of some concepts and strategies.

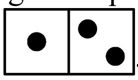
The advantages are:

1. Enjoyable way to reinforce concepts which would be require dull drill and practice exercises.
2. Develop a positive attitude towards mathematics as an enjoyable subject by avoiding consolidation through exercises out of context.
3. Develop problem solving strategies (what is my best move, is there a winning strategy for one or for both players, how many different moves are possible, what are the chances of winning, etc.).
4. Active involvement of ALL pupils.

Section B1: Dominoes

The domino game is a game that can be used to consolidate a wide range of skills, among them solving of linear equations.

General information on Dominoes

Dominoes are formed by joining two squares side by side to make a rectangular shape like this one . Illustrated is a 1 - 2 domino. A complete set of dominoes consists of 28 dominoes with all possible combinations of zero (blank square) to six spots, including ‘doubles’, i.e., a zero - zero domino (double blank), a one - one domino, etc. It is a game for 2 - 4 players. Players take turns to match a domino to one end of the line of dominoes or, if this is not possible, to draw a domino from the pile. In more detail it is played according to the following rules:

1. Turn the dominoes face down.
2. Each player is given 5 dominoes and can open his/her dominoes.
3. Open one of the face down dominoes as a starter.
4. Decide who starts to play.
5. (a) In turn the players place a matching domino at either end of the domino on the table, forming a line of dominoes.
5. (b) When a player forms a match by placing a double-domino, he or she places it at right angles to the end of the line of dominoes as if crossing a “T”. This forms a branch in the line, and creates two “ends”, instead of one, where succeeding dominoes may be placed.
6. If a player cannot find a matching domino she/he picks up another domino from those left face down on the table and misses his/her go. If all dominoes from the table are finished, the player simply misses a go.
7. The player to finish his/her dominoes first wins the game or if nobody can play any more and nobody has exhausted all his/her dominoes, the winner is the player with the least number of dominoes left.

The ‘traditional’ domino game can be adapted to cover a wide variety of topics in the mathematics syllabus. Here you will use it to consolidate solving of linear equations.

The linear equation version of Dominoes

The game is played by the above rules. However, instead of dots, each domino has two **equations** printed on it. For example, the 1 – 2 domino above might have $2a + 6 = 8$ on the left side, and $16 - 2t = 12$ on the other side. Note that a 1 – 1 domino can and should have two **different** equations on it, though they both evaluate to 1.

Method for making dominoes.

1. Choose the number of alternatives you want to use, thus determining the size of the pack. The standard pack of dominoes has seven alternatives, shown here as A through G.

Table 1

						28	36	45	55
AA	BB	CC	DD	EE	FF	GG	HH	II	JJ
AB	BC	CD	DE	EF	FG	GH	HI	IJ	
AC	BD	CE	DF	EG	FH	FI	HJ		
AD	BE	CF	DG	EH	FI	GJ			
AE	BF	CG	DH	EI	FJ				
AF	BG	CH	DI	EJ					
AG	BH	CI	DJ						
AH	BI	CJ							
AI	BJ								
AJ									

For a set with 10 alternatives you have to make the above 55 dominoes. For a set with 7 alternatives you need 28 dominoes, etc.

2. List the variations you are going to use. For a set with 8 alternatives (36 dominoes) you will need 9 variations for each (because of the double domino).

For example if you make a domino set for solving linear equations of the format $ax + b = c$ or $p + qx = r$; a, b, c, p, q and r being integers and the equations have as solutions $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, 3, 4, 5, or 6, you have to make 9 different equations with solution $\frac{1}{2}$, 9 with solution 1, etc.

One equation for each answer is tabulated in table 2. For completeness, each answer needs eight more equations.

Table 2

A: answer $\frac{1}{2}$	$2a + 6 = 7$	B: answer 1	$7b - 1 = 6$
C: answer $1\frac{1}{2}$	$3a + 1.5 = 6$	D: answer 2	$3a - 1 = 5$
E: answer 3	$2x + 7 = 13$	F: answer 4	$13 - 3t = 1$
G: answer 5	$22 - 2t = 12$	H: answer 6	$17 - p = 11$

3. As you make a domino, cross out the combination (from table 1) and the value variation (from table 2) you have used.

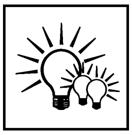
Complete table 2 for the set 1. Check that the set is a proper domino set with 36 dominoes.

On the next pages you find the following sets:

Set 1: Format $ax + b = c$ or $p + qx = r$; a, b, c, p, q and r being integers, the equations have as solutions $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, 3, 4, 5, or 6

Set 2: Format $ax + b = c$ or $p + qx = r$; a, b, c, p, q and r being integers, the equations have as solutions $^{-}1$, $^{-}2$, $^{-}3$, $^{-}4$, $^{-}5$, 2, 3 or 4

Set 3: Format $a(px + q) = b$ or $c(m + nx) = d$; a, b, c, d, p, q, m, n being integers, the equations have as solution $\frac{1}{2}$, 1, 2, 3, 4, 5, or 6



Unit 4, Practice task

1. Play "Dominoes" in your class with the sets provided. Write an evaluative report.
2. Make and use another set of dominoes yourself to consolidate the type of linear equations needed with your class or make a 'mixed' set for higher achievers.

Present your assignment to your supervisor or study group for discussion.

Domino set 1: solving linear equations

$2a + 6 = 7$	$10 - 4a = 8$	$4a - 1 = 5$	$2b + 4 = 7$
$4a - 2 = 0$	$12 - 4a = 6$	$7b - 1 = 6$	$12 - 5c = 7$
$6a + 5 = 8$	$5a + 2 = 7$	$d + 9 = 10$	$3a - 1 = 5$
$8 + 2a = 9$	$4d + 2 = 10$	$8 - y = 7$	$2x + 7 = 13$
$10 + 4a = 12$	$2a + 1 = 7$	$3d + 5 = 8$	$2b + 7 = 15$
$7 - 2a = 6$	$4q + 2 = 18$	$10 - 5z = 5$	$3x - 8 = 7$
$8 - 4a = 6$	$6b + 2 = 32$	$6 + 3y = 9$	$3p - 12 = 6$
$6x + 10 = 13$	$7c - 20 = 2$	$^-4c + 6 = 2$	$6a - 9 = 0$
$4y + 7 = 15$	$5b + 3 = 13$	$4c + 1 = 13$	$5p + 4 = 19$

$19 - 4d = 3$	$3t + 7 = 19$	$17 - 3s = 2$	$2f + 11 = 21$
$3h + 2 = 20$	$10 - q = 4$	$2d - 1 = 5$	$6r - 7 = 17$
$12x + 6 = 30$	$3z - 1 = 8$	$7 - 2x = 1$	$3a - 10 = 5$
$7x - 3 = 11$	$5c + 3 = 23$	$13 - 4r = 1$	$12 - 2y = 0$
$10 - 4d = 2$	$4c - 8 = 12$	$19 - 5b = 4$	$4s + 4 = 10$
$15 - 4t = 7$	$4d - 7 = 17$	$4k - 3 = 13$	$4r + 7 = 27$
$30 - 12w = 6$	$2e - 1 = 2$	$12 - 2e = 4$	$5a - 8 = 22$
$12 - c = 7$	$17 - p = 11$	$13 - 3c = 1$	$2x + 8 = 11$
$22 - 2t = 12$	$3a + 1.5 = 6$	$19 - 2g = 7$	$6 - 2i = 3$

Domino set 2: solving linear equations

$4a + 10 = 2$	$3p + 5 = -1$	$4a + 18 = 2$	$1 - 3p = 13$
$2a + 7 = 1$	$4 - 5b = 19$	$4a - 10 = -2$	$3a + 5 = 11$
$6b + 32 = 2$	$2 - 3c = 17$	$5a + 7 = 2$	$2 - 4c = 6$
$2a - 7 = -1$	$3z - 8 = 1$	$4a - 18 = -2$	$3c - 14 = -2$
$4t + 15 = 7$	$2x + 13 = 7$	$7d + 11 = -3$	$3r + 7 = -8$
$5b + 13 = 3$	$2m + 15 = 7$	$7 - 4x = 15$	$7u + 6 = -1$
$12w + 30 = 6$	$4y - 15 = -7$	$6 - 12d = 30$	$4d - 13 = 3$
$2 - 4a = 10$	$2s - 13 = -7$	$5t + 23 = 3$	$12x - 30 = -6$
$4r + 13 = 1$	$3y + 19 = 7$	$6e + 17 = -7$	$4e + 12 = -8$

$5t + 19 = 4$	$5b - 13 = {}^-3$	$4a + 13 = {}^-3$	$i + 10 = 9$
$3k + 8 = {}^-1$	$2s + 21 = 11$	$4 - 2j = 12$	$12 - 3k = 0$
$2d + 5 = {}^-1$	$7 - 5c = 12$	$3 - 4t = 19$	$5g - 19 = {}^-4$
$1 - 2r = 7$	$4 - 4h = {}^-4$	$3a + 5 = {}^-10$	$7c - 11 = 3$
$1 - 4g = 13$	$4f - 13 = {}^-1$	$4r + 27 = 7$	$7 - n = 8$
$3s + 8 = 5$	$4 + 4t = 12$	$7 - v = 12$	$2v - 15 = {}^-7$
$5 - 5f = 10$	$3r - 19 = {}^-7$	$12 - 2u = 22$	$2w - 5 = 1$
$9 + 3y = 6$	$1 - 2c = {}^-5$	$7 - 4h = {}^-1$	$5t - 23 = {}^-3$
$6c - 17 = {}^-7$	$4b - 15 = {}^-3$	$2x + 7 = 11$	$3 - 2p = {}^-3$

Domino set 3: solving linear equations

$2(x + 2) = 8$	$3(p + 2) =12$	$3(8 – c) = 9$	$2(2a + 1) =22$
$4(4e + 1) = 12$	$6(2d + 3) = 24$	$2(2w – 1) = 10$	$6(8 – 2p) =12$
$4(5a – 3) = 8$	$5(9 – 3s) = 30$	$5(x + 4) = 40$	$5(12 – 2e) = 20$
$3(10 – 2k) = 18$	$4(2f + 1) = 28$	$8(12 – b) = 48$	$3(5 + q) = 33$
$2(2z + 4) = 16$	$2(2p + 3) = 22$	$2(2g – 3) =24$	$6(3a – 5) = 24$
$4(3c + 2) = 32$	$6(u + 1) = 42$	$3(10 – 2y) = 0$	$3(j + 7) = 33$
$3(4i – 1) = 21$	$5(4s – 1) = 5$	$2(2 + x) = 14$	$2(3 + 2z) = 20$
$2(2v + 1) = 10$	$5(3e – 4) = 55$	$3(2 + 2c) = 36$	$3(2w – 1) = 0$
$4(2d + 1) = 20$	$3(5f + 2) = 21$	$4(3p – 7) = 32$	$7(5t – 5) = 0$

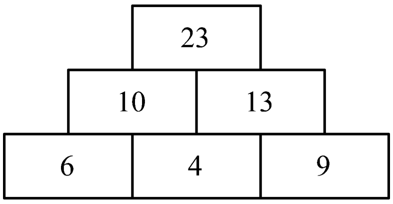
$6(u + 4) = 42$	$2(3e + 10) = 44$	$3(2r + 1) = 27$	$4(1 + 2b) = 52$
$3(7 - 2c) = 3$	$5(9 - f) = 15$	$6(p - 4) = 0$	$2(8 - 2x) = 14$
$8(9 - 2k) = 24$	$4(6 - 2s) = 20$	$2(2q - 1) = 14$	$3(6 - 3z) = 9$
$8(7 - 2v) = 8$	$2(5 - j) = 8$	$3(6t + 1) = 12$	$2(2c - 8) = 8$
$4(5f + 1) =14$	$3(2x + 5) = 21$	$4(7 - 2x) = 20$	$3(p - 4) = 6$
$15 - 4t = 7$	$4d - 7 = 17$	$4k - 3 = 13$	$4r + 7 = 27$
$30 - 12w = 6$	$2e - 1 = 2$	$12 - 2e = 4$	$5a - 8 = 22$
$12 - c = 7$	$17 - p = 11$	$13 - 3c = 1$	$2x + 8 = 11$
$22 - 2t = 12$	$3a + 1.5 = 6$	$19 - 2g = 7$	$6 - 2i = 3$

Section B2: Pyramids

The goal of the following outline of an activity is to provide a context within which pupils can gain experience in forming linear equations and solving them.

Step 1: Explain how number pyramids are built.

A pyramid is built such that each number in layers above the first is the sum of the two numbers immediately below it.



Starting from the given bottom numbers 6, 4 and 9, the next layer is filled by adding

6 + 4 = 10, and 4 + 9 = 13. The top number is 10 + 13 = 23.

Step 2: Pupils complete pyramids using numbers only.

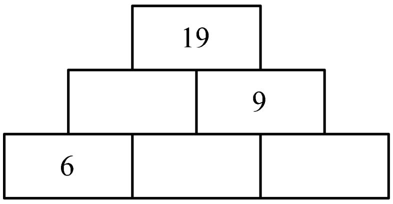
The difficulty level can be controlled by:

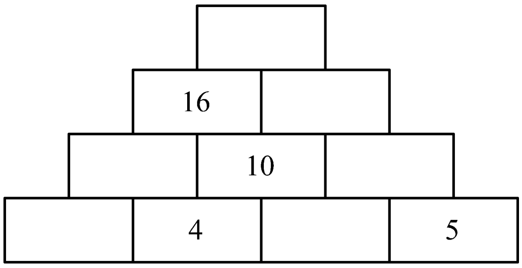
(i) increasing the number of layers

(ii) the choice of numbers (magnitude of the numbers given, decimals, fractions, integers)

(iii) placing the given numbers at various layers, i.e., not all in the bottom layer.

Here are two examples of pyramids to be completed

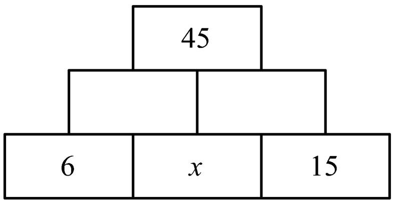


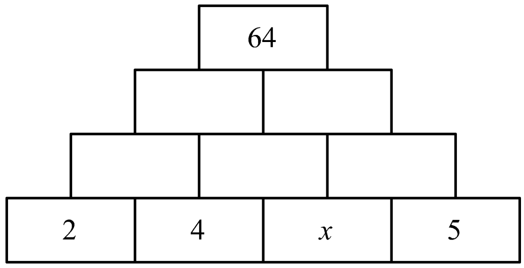


Step 3: A variable is introduced.

A variable is used in one of the building blocks. Completing the pyramid will now lead to an equation. The value of the variable is to be found (solving the equation).

Here two examples of pyramids with variables.





Again the difficulty can be controlled. The root of the equation can be ensured to be a whole number, a fraction, an integer, etc. The given values need not all to be placed at the bottom layer.

Unit 4: Practice task

1. Try out each of the different techniques for solving equations in class with pupils.

   (a) cover-up techniques

   (b) ‘undoing’ working backwards using flow diagrams

   (c) trial and error technique

   (d) performing the same operation on both sides (balance model), which could move into transposing technique (“change side - change sign”)

   (e) graphical techniques

   (f) using algebra tiles

   Compare the outcomes. Which technique(s) help(s) pupils most in their understanding?

   Justify your choice. What are pupils’ reactions to the different techniques?

   Write an evaluative report.

   In your report include a mathematical comparison of the various techniques. Which technique(s) is/are generally applicable? What are the limitations of each technique? What are the disadvantages / advantages of each technique? Which technique(s) will enhance most the understanding of pupils?

2. Develop worksheets for the ‘pyramid’ activity with various levels of difficulty from ‘straightforward’ to challenging and investigative in nature.

   Try the worksheets out in your class and write an evaluative report.

Present your assignment to your supervisor or study group for discussion.

Summary

For teaching how to solve equations, this unit has stressed:

- the use of multiple solving methods, especially structural methods
- the use of manipulatives
- the use of games

The overall aim is to ensure that all your students gain a strong grasp of what equation solving **really means** and when it is **appropriate** to **use**.

In the long run, of course, it is not necessary for every student to retain and use every method. Your goal should be that every student, using his or her preferred few methods, can tackle every “real” problem that lends itself to solving via a linear equation of a variable.

Unit 2: Answers to Self mark exercises

Self mark exercise 1

1. Covering $3(x - 4)$ gives for $3(x - 4)$ the value 18, and for $x - 4$ the value 6, so $x = 10$.

2. $x = 4\frac{1}{6}$ 3. $x = 6$ or $x = 2$ 4. $x = 5\frac{1}{2}$

5. $x = 5$ 6. $x = 22$

7. $x = 2\sqrt{2}$ or $x = {}^{-}2\sqrt{2}$ 8. $x = 10$ or $x = 14$

Self mark exercise 2

I. Equations are the same as in self mark exercise 6. An illustration of the working follows here:

4.

$$x \xrightarrow{\text{Subtr from 6}} 6 - \times \xrightarrow{\text{reciprocal}} \frac{1}{6-x} \xrightarrow{\times 4} \frac{4}{6-x} \xrightarrow{-3} \frac{4}{6-x} - 3 = 5$$

Reversing the flow

$$5\frac{1}{2} \xleftarrow{\text{Subtr from 6}} \frac{1}{2} \xleftarrow{\text{reciprocal}} 2 \xleftarrow{\div 4} 8 \xleftarrow{+3} 5$$

8.

$$x \xrightarrow{\text{Subtr from 12}} 12 - x \xrightarrow{\text{square}} (12 - x)2 = 5$$

Reversing the flow

$$10 \xleftarrow{\text{Subtr from 12}} 2$$
$$14 \xleftarrow{\text{Subtr from 12}} {}^{-}2$$
$$2,\ {}^{-}2 \xleftarrow{\text{square roots}} 4$$

II. The flow chart method allows starting with the variable in one place only. If the variable appears more than once it cannot be modelled in a linear flow chart.

Self mark exercise 3

1. 2.54 2. 2.28, -2.28 3. 3.39

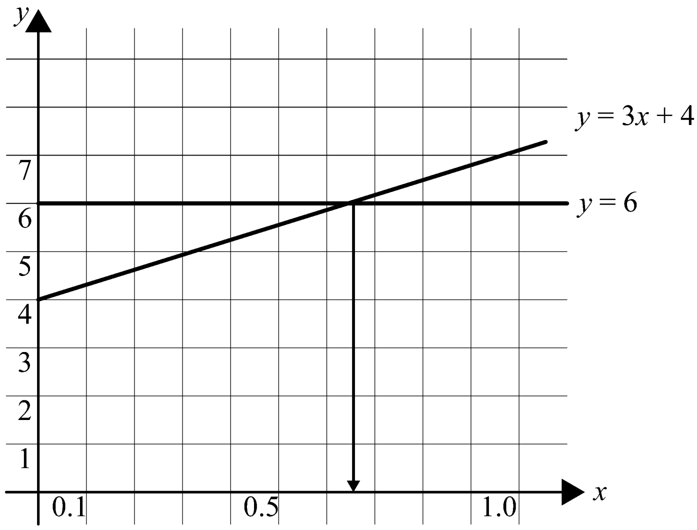
Self mark exercise 4

1. 7.5 2. all real numbers (identity) 3. 11 4. 0

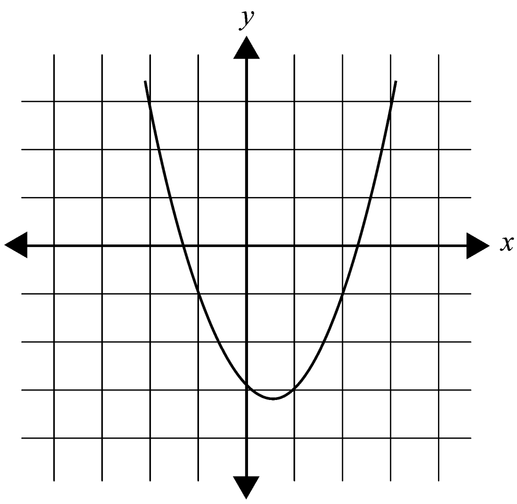
5. no real solution (false equation) 6. all real numbers (identity)

Self mark exercise 5

1. Scale used does not allow great accuracy.
   Better scales illustrated below. $x = 0.67$ (2dp)

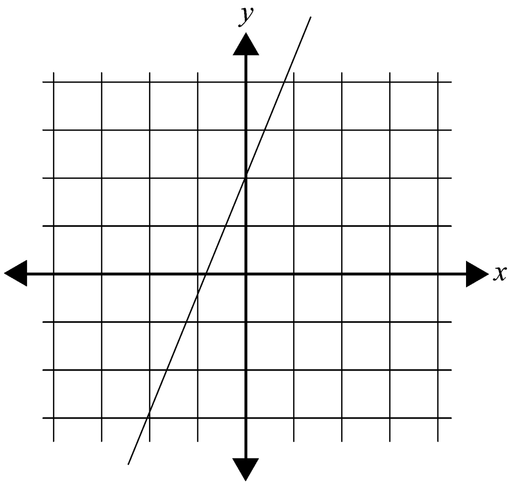


2.



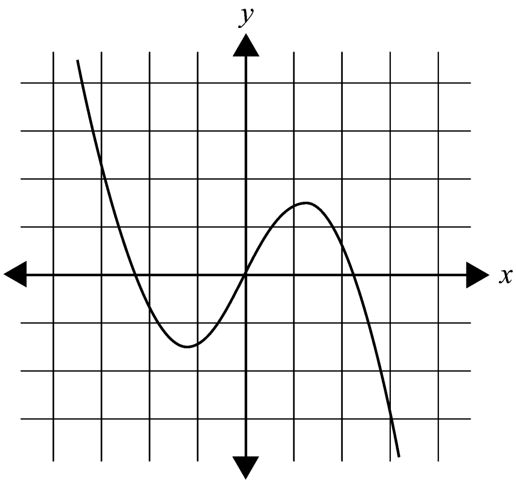
Solutions: $x \approx {}^{-}1.3$ or 2.3

3.



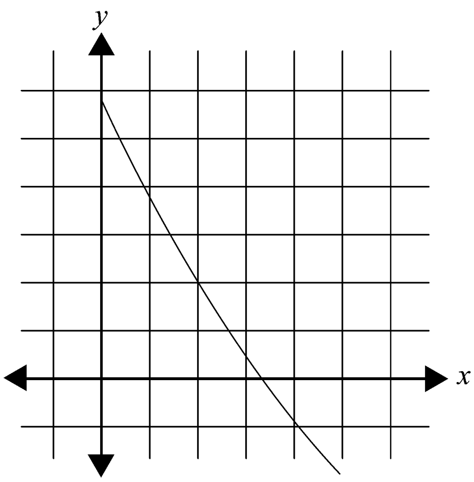
Solution: $x \approx {}^{-}0.8$

4.



Solution: $x \approx {}^{-}2.3, 2.3$

5.



Solution: $x \approx 3.4$

Self mark exercise 6

Placing of the tiles similar to the examples.

Unit 3: Solving quadratic equations

Introduction to Unit 3

Many relationship are not linear, for example the distance *d* metres travelled by a falling object in a time *t* seconds or the height *h* m of a ball above the ground *d* m from the person that kicked the ball. These relationships are quadratic. The first known solution to a quadratic equation is found in a papyrus from Ancient Egypt about 2000 BC. From around the same time clay tablets indicate that the Babylonians were also familiar with the quadratic formula. The Greeks solved quadratic equations by geometrical methods: completing a square.

Purpose of Unit 3

The main aim of this unit is to look at techniques of solving quadratic equations that follow a structural approach, as opposed to an algorithmic approach, i.e., applying a formula that might be ill understood. Pupils at times do know the 'abc-formula' but are unable to explain or derive the formula. The starting point is modelling real life situations leading to quadratic equations, i.e., creating a need to solve equations. In this unit you will look at quadratic equations and techniques to solve them. The emphasis will be: how can you assist pupils in acquiring relational knowledge about solving of quadratic equations.

Objectives

When you have completed this unit you should be able to:

- state the meaning of $\sqrt{\ }$ symbol
- state pupils' difficulties with square roots
- set questions to pupils to create the need to learn techniques to solve quadratic equations
- solve quadratic equations by using
  - factorisation
  - graphical method
  - completing the square
  - using the formula
  - trial and improvement
- set and monitor activities for pupils related to acquiring knowledge about quadratic equations
- use teaching techniques to assist pupils to overcome problems in the learning of solving of quadratic equations

Time

To study this unit will take you about 12 hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topic.

Unit 3: Solving quadratic equations

Section A: Square roots

What is a square root?

There is confusion among teachers and hence among pupils related to square roots. What is $\sqrt{9}$? Is it 3? Or is it 3 and $^{-}3$, as both 3 and $^{-}3$ when squared give 9? As in many cases in mathematics, one must be aware of (i) definitions that the author might use and (ii) conventions agreed upon within the mathematical community.

Before continuing reflect on your own classroom practice and write down your response to each of the following questions.

1. What do you consider the correct response to the value of $\sqrt{9}$? Justify your answer.
2. What is the complete working you initially expect from pupils as a solution of each of the following equations?

   (i) $x^2 = 16$ (ii) $\sqrt{16} = x$

Refer to your notes when you read through the next section.

Definitions as to what is the square root of a number differ. Not all books use the same definition. You might meet the following two different definitions.

Definition 1

The square root of a number N is the number which when squared gives N.

Hence if N is a positive number each number will have two square roots, a positive and a negative one.

The square roots of 16 are 4 and $^{-}4$, as both 4 and $^{-}4$ when squared give 16. Taking the square root is seen as the inverse operation of squaring.

Negative numbers do not have square roots (but do have cube roots, fifth roots, etc.).

The notation used for the two square roots of a positive number N:
$\sqrt{N}$ and $^{-}\sqrt{N}$

Definition 2

The square root of any positive number N is the positive number which when squared gives N. Notation used is $\sqrt{N}$. With this definition each positive number has only one square root and taking the square root is NOT the inverse operation of squaring.

It is advised to use the second definition to avoid confusion between the question in words and in symbols following from the first definition.

Following the first definition you are to respond to the question:

"What is (are) the square root(s) of 16? " (word format) with 4 and $^-4$ and to the question "What is $\sqrt{16}$?" (symbol format) the response is 4.

Using the second definition the responses to both questions are the same, i.e., 4, which might avoid confusion in the mind of pupils.

The unique meaning of the $\sqrt{\ }$ *symbol*

Whatever definition the teacher agrees with pupils to use, in BOTH cases, the $\sqrt{\ }$ symbol is standing for the **positive** number which when squared gives the number under the root symbol.

Hence $\sqrt{16} = 4$ and NOT $^-4$.

Working of $x^2 = 16$ *and* $\sqrt{16} = x$.

There is frequently confusion between the workings of $x^2 = 16$ and $\sqrt{16} = x$.

The first is a quadratic equation and a solution could look like this, using factorisation:

$(x^2 - 16) = 0$

$(x - 4)(x + 4) = 0$

$x - 4 = 0$ or $x + 4 = 0$

$x = 4$ or $x = {}^-4$

or a shortcut—to be discouraged for use by pupils as it induces errors—is

$x^2 = 16$

$x = {}^+\sqrt{16}$ or $x = {}^-\sqrt{16}$

$x = 4$ or $x = {}^-4$

The common **error presentation** being:

$x^2 = 16$

$x = \sqrt{16}$

$x = 4$ or $x = {}^-4$

Due to the meaning of $\sqrt{\ }$ the value of $\sqrt{16}$ is 4 only (and NOT $^-4$), hence the above is an incorrect working.

Alternative **correct working** (but often misunderstood why it is correct):

$$x^2 = 16$$
$$\sqrt{x^2} = \sqrt{16}$$
$$x = 4 \text{ or } x = {}^-4$$

This is strictly mathematically a correct working but misunderstood by most pupils who explain the two answers in the last line by saying (erroneously!) that $\sqrt{16}$ is 4 or $^-4$, while the 'correctness' of the working is due to the fact that the left hand side of the equation has two possible outcomes: $\sqrt{x^2} = x$ (when $x \geq 0$) or ^-x (when $x \leq 0$), an in-between step not shown. If shown the working would look like:

$$x^2 = 16$$
$$\sqrt{x^2} = \sqrt{16}$$
$$x = 4 \text{ or } {}^-x = 4$$
$$x = 4 \text{ or } x = {}^-4$$

To avoid the above complications and ill understood relationships such as $\sqrt{a^2} = a$ or ^-a, $\sqrt{a^4} = a^2$ (and NOT $^-a^2$), $\sqrt{a^6} = a^3$ or $^-a^3$, it is advised, whenever possible, to solve quadratic equations of the format $x^2 = N$ by factorisation as illustrated above.

Self mark exercise 1

1. How would you explain to pupils in form 3 the following?

$$\sqrt{a^2} = a \text{ or } {}^-a$$
$$\sqrt{a^4} = a^2 \text{ (and NOT } {}^-a^2\text{)}$$
$$\sqrt{a^6} = a^3 \text{ or } {}^-a^3$$

Check your answers at the end of this unit.

In this section you are going to work on techniques of solving quadratic equations. Before a technique can be introduced pupils must feel a need for it (Section E1). This was discussed in the previous section and is obviously applicable here as well.

The techniques for solving quadratic equations you are to look at are:

Section B2 Solving by factorisation

Section B3 Solving by graphical method

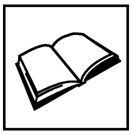
Section B4 solving by completing the square

Section B5 Solving by using the quadratic formula

Section B6 Solving by trial and improvement method

Although techniques can be practised in isolation, i.e., without a context, the need to practise the technique must be clear to pupils: they want to answer certain realistic questions in which the solving of a quadratic equations is involved. At the same time, even when practising the techniques, as a clear reminder why it is be done, some context questions should be included. Mathematics, at this level, is to be a tool for pupils to answer realistic questions. Few will appreciate the abstract structure of the system.

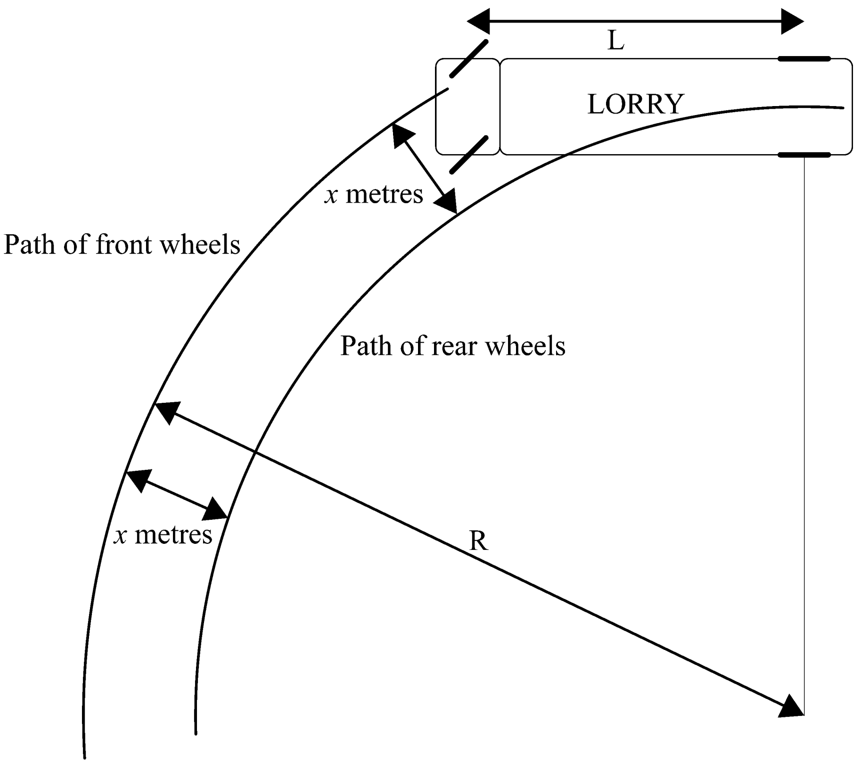
Section B1: Creating the need for solving quadratic equations



Before moving into techniques for solving quadratics create a need for it. Present pupils with a number of situations all leading to quadratic equations. Form the equation prior to solving it. Here are a few suggestions for questions leading to quadratic equations.

The turning lorry

A lorry with a wheel base of L metres is turning a left hand corner. The front wheels are describing a circle with radius R metres. The rear wheels are 'cutting the corner' by a distance of *x* metres.



- Form an equation involving x, L and R.
- A lorry with a wheel base of 4 m turns a corner of radius 5 m. By how much do the rear wheels cut the corner?

Making boxes

An open box, holding 1000 cm^3, is to be made. The rectangular base is to have sides of length x cm and $(x + 8)$ cm. The height of the box is 10 cm. What is the length and width of the rectangular base of the box?

Another open box, also holding 1000 cm^3, is to have a rectangular base with the length 10 cm more than the width. The height is to be 12 cm. What is the length and the width of the rectangular base of the box?

Self mark exercise 2

1. Form the equations to solve the above problems.

Check your answers at the end of this unit.

Section B2: Solving quadratic equations by factorisation

To solve a quadratic equation of the form $x^2 + px + q$

A prescription:

Step 1. Try to factorise the expression by looking for two numbers with the sum p and the product q or by using any other valid technique (e.g., using algebra tiles).

Step 2. Equate the factors of the factorised equation equal to zero.

Step 3. Solve the linear factor equations.

Step 4. Check your answers.

Example

Solve for x in the equation $x^2 - 7x - 18 = 0$

Step 1: To factorise you try to find two numbers with the sum $^-7$ and the product $^-18$. Trial and error gives $^-9$ and $^+2$.

$x^2 - 7x - 18 = 0$ $\quad (x - 9)(x + 2) = 0$ (factorising the LHS)

Step 2: $\quad x - 9 = 0$ or $x + 2 = 0$ (equating factors to zero)

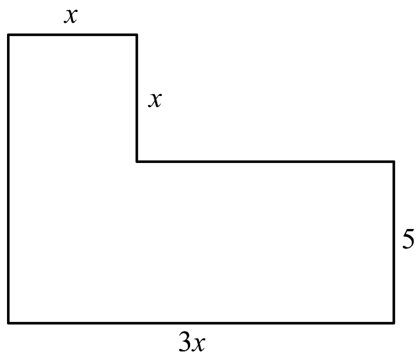
Step 3: $\quad x = 9$ or $x = {}^-2$ (solving the factor equations)

Step 4: Checking
$x = 9$ LHS = $(9)^2 - 7(9) - 18 = 81 - 63 - 18 = 0$ = RHS (correct)
Checking
$x = {}^-2$ LHS = $({}^-2)^2 - 7({}^-2) - 18 = 4 + 14 - 18 = 0$ = RHS (correct)

Self mark exercise 3

Solve the equations in question 1 - 7.

1. $p^2 - 10p - 24 = 0$
2. $q^2 - 10q + 24 = 0$
3. $r^2 + 10r - 24 = 0$
4. $t^2 - 14t + 24 = 0$
5. $s^2 - 5s - 24 = 0$
6. $u^2 - 23u - 24 = 0$
7. $v^2 - 25 = 0$
8. The height of a triangle is $2x$ cm and the corresponding base is $(7 - x)$ cm long. If the area enclosed by the triangle is 12 cm^2, find height and base of the triangle.
9. A rectangle encloses an area of 20 cm^2. Its sides are of length x cm and $(x - 1)$ cm. Find the length of the sides.
10. The product of a number and three less than the number is 54. What is the number?
11. An open box has a square base with sides of length x cm and a height of 10 cm. If the total surface area of the box is 225 cm^2, find the length of the side of the square base of the box.
12. The area enclosed by the shape is 54 cm^2. Find the length of each of the sides of the shape.



13. The product of two consecutive odd numbers is 483. Find the numbers.
14. A man walks due East and then South the same distance plus 7 km. He is now 17 km from his starting point. How far East did he walk first?
15. a) For which value(s) of p will the equation $x^2 = p$ have (i) two (ii) one (iii) no solution?

    b) What about the equation $(x + a)^2 = p$?
16. True or False: "If you cannot factorise $x^2 + px + q = 0$ then there are no solutions to the equations".

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17. Nkamogelang says "All quadratic equations do have a solution, but the solution is not always a whole number". Is she correct? Investigate.
18. Lerothodi solved the equation $(x + 4)(x - 5) = 20$

    and wrote $x + 4 = 20$ or $x - 5 = 20$

    $x = 20 - 4$ or $x = 20 + 5$

    $x = 16$ or $x = 25$

    Discuss Lerothodi's method.

    Describe in details the 4 remedial steps (diagnosing the error, creating conflict in the mind of the pupil, resolving the conflict by having pupil construct the correct algorithm, consolidation of correct algorithm) you would take to assist the pupil.

Check your answers at the end of this unit.

Unit 3: Practice task

1. Describe in detail the activity you would set to pupils to solve quadratic equations of the form $ax^2 + bx + c = 0$ using the factorisation method.

   Develop the consolidation exercise, taking into account the different levels of understanding of pupils.

Present your assignment to your supervisor or study group for discussion.

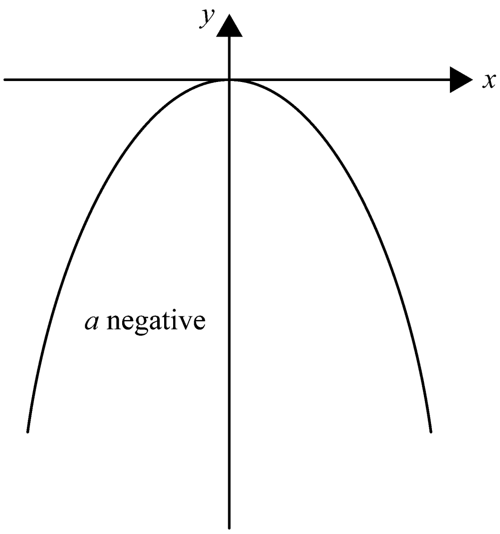
Section B3: Graphs of quadratic functions

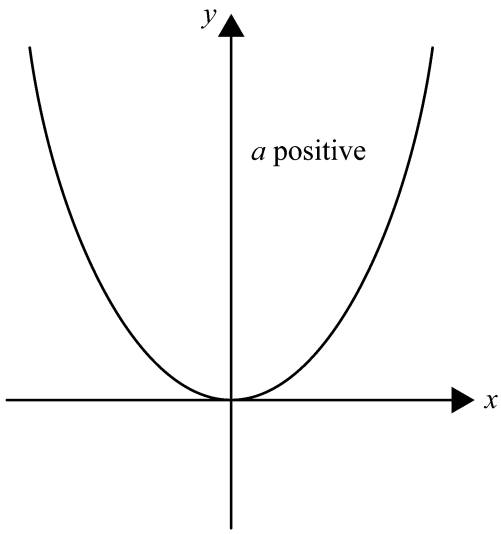
The solving of equations using a graphical approach was discussed previously. In order to solve quadratic equations graphically, knowledge about the basic facts of the graphs of $y = ax^2 + bx + c$ is extremely useful, although not necessarily a prerequisite for pupils (for the teacher it obviously is).

This section therefore first revisits some of the basic facts of quadratic graphs. These facts are presented here as a reminder to the teacher. The covering of graphs of quadratic functions with pupils—using a discovery approach—is not discussed in this section.

I Graphs with equations $y = ax^2$

The graph of the **quadratic** relation $y = ax^2$ is called a **parabola**.





Each parabola has either a lowest point (when a is positive) or a highest point (when a is negative).

The highest / lowest point is called the **vertex** or **turning point** of the parabola.

Each parabola has a vertical line of symmetry also called the axis (of symmetry) of the parabola.

For parabolas with equation $y = ax^2$, the vertex is the point with coordinates (0, 0), i.e., the origin.

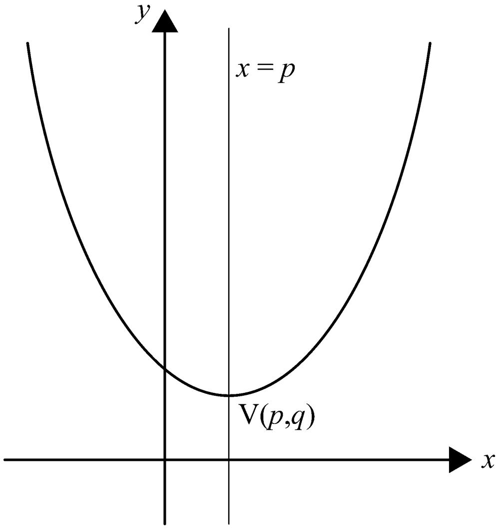
The axis of symmetry is the y-axis with equation $x = 0$.

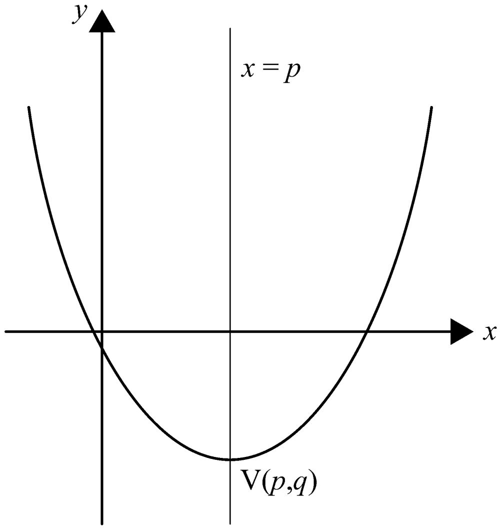
II Graphs with equations $y = a(x - p)^2 + q$.

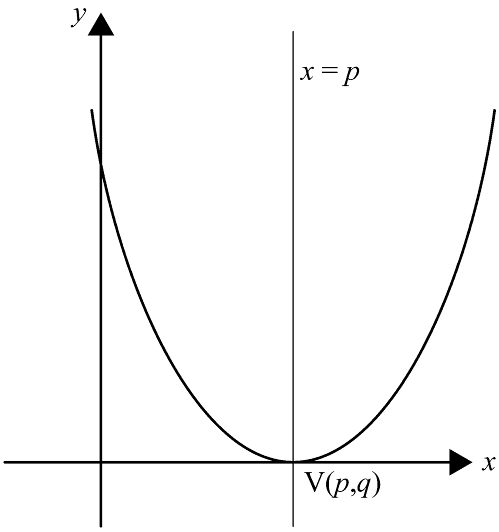
The graph with equation $y = a(x - p)^2 + q$ if $a > 0$

The graph has a minimum q when $x = p$. The coordinates of the vertex are (p, q).

Axis of symmetry has the equation $x = p$.







With respect to the axes the graph can be (i) completely above the x-axis (ii) cutting the x-axis in two distinct points (iii) touching the x-axis.

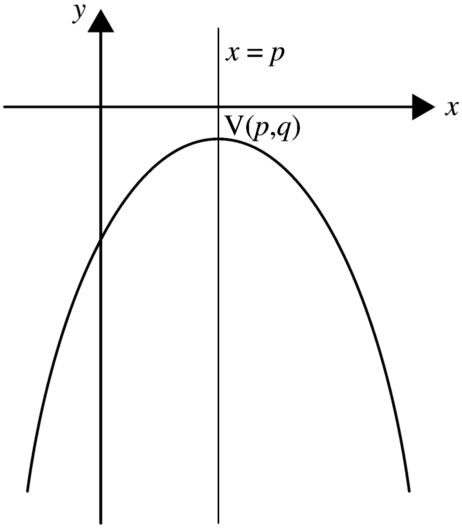
The graph with equation $y = a(x - p)^2 + q$ if $a < 0$

The graph has a maximum q when $x = p$. The coordinates of the vertex are (p, q).

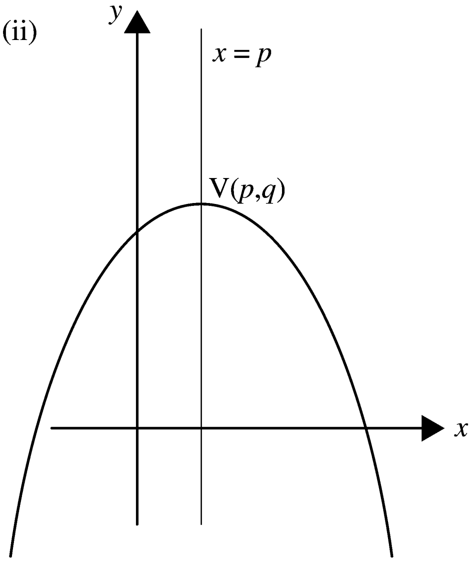
Axis of symmetry has the equation $x = p$.

With respect to the axes the graph can be (i) completely below the x-axis (ii) cutting the x-axis in two distinct points (iii) touching the x-axis.
This is illustrated below in sketch graphs.

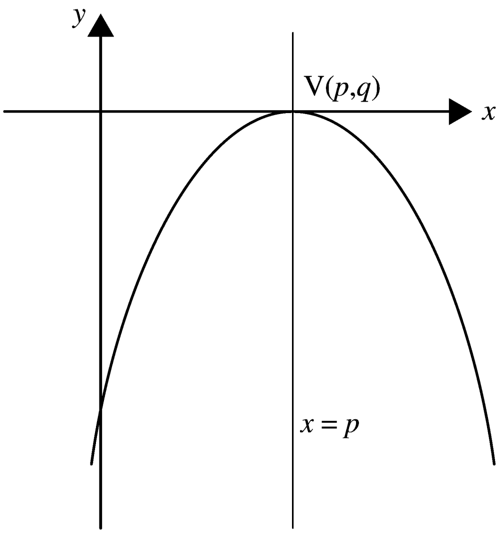
(i)



(ii)



(iii)



For pupils to discover the above listed facts you are to guide them through various steps.

Here is a possible outline:

Step 1. Investigating the graphs with equation $y = ax^2$

Pupils plot graphs with equations $y = ax^2$ for $a = \frac{1}{2}$, 1, 2 and $a = {}^{-}\frac{1}{2}$, $^{-}1$, $^{-}2$.

They start with the case $a = 1$ then the other values of a on the same axes system.

Through guided questions they are to conjecture:

The graph with equation $y = ax^2$ (i) has a minimum for a > 0 and a maximum for a < 0 (ii) the graphs have a line of symmetry, the y-axis, with equation $x = 0$ (iii) increasing positive values of a make the 'opening' of the curve at a fixed y-value narrower (same for decreasing negative values of a).

Step 2. Graphs of quadratics with equation $y = ax^2 + q$ are investigated.

Pupils plot on the same axes (for a sequence) graphs with equations

(i) $y = x^2$, $y = x^2 - 2$, $y = x^2 - 1$, $y = x^2 + 1$, $y = x^2 + 2$

(ii) $y = 2x^2$, $y = 2x^2 - 2$, $y = 2x^2 - 1$, $y = 2x^2 + 1$, $y = 2x^2 + 2$

(iii) $y = {}^-x^2$, $y = {}^-x^2 - 2$, $y = {}^-x^2 - 1$, $y = {}^-x^2 + 1$, $y = {}^-x^2 + 2$

and more of these sequences if needed.

Comparing the graphs and with guided questions, asking for the coordinates of the vertex and comparing with the equation of the graph, pupils are to discover that the vertex of the graph with equation $y = ax^2 + q$ is $V(0,q)$.

Step 3. Pupils investigate graphs with equation $y = a(x - p)^2$

Using the same axis (for a sequence) graphs are plotted with equations

(i) $y = x^2$, $y = (x - 2)^2$, $y = (x - 1)^2$, $y = (x + 1)^2$, $y = (x + 2)^2$

(ii) $y = \frac{1}{2}x^2$, $y = \frac{1}{2}(x - 2)^2$, $y = \frac{1}{2}(x - 1)^2$, $y = \frac{1}{2}(x + 1)^2$, $y = \frac{1}{2}(x + 2)^2$

(iii) $y = 2x^2$, $y = 2(x - 2)^2$, $y = 2(x - 1)^2$, $y = 2(x + 1)^2$, $y = 2(x + 2)^2$

(iv) $y = {}^-x^2$, $y = {}^-(x - 2)^2$, $y = {}^-(x - 1)^2$, $y = {}^-(x + 1)^2$, $y = {}^-(x + 2)^2$

(v) $y = {}^-2x^2$, $y = {}^-2(x - 2)^2$, $y = {}^-2(x - 1)^2$, $y = {}^-2(x + 1)^2$, $y = {}^-2(x + 2)^2$

and more of these sequences depending on the level of understanding of the pupils (some pupils will need to work through more sets than others).

Comparing the graphs in each sequence pupils are to relate the equation of the line of symmetry (or axis) to the equation of the graph. The expected outcome is that pupils will conjecture that the line of symmetry of the graph with equation $y = a(x - p)^2$ is $x = p$.

Step 4. The work in the previous step is now joined in the investigation of the graphs with equation $y = a(x - p)^2 + q$.

Pupils are again required to plot graphs on the same axis with equations (for example)

(i) $y = x^2$, $y = (x - 2)^2 + 1$, $y = (x - 1)^2 - 2$, $y = (x + 1)^2 + 2$, $y = (x + 2)^2 - 3$

and similar sequences.

Writing down the coordinates of the vertex and the equation of the line of symmetry, pupils are expected to come up with the conjecture:

The graph with equation $y = a(x - p)^2 + q$

(i) If $a > 0$

The graph has a minimum q when $x = p$. The coordinates of the vertex are (p, q).

Axis of symmetry has the equation $x = p$.

(ii) If $a < 0$

The graph has a maximum q when $x = p$. The coordinates of the vertex are (p, q).

Axis of symmetry has the equation $x = p$.

The work in the previous investigative steps is to be consolidated through an exercise covering two types of questions.

Type 1: Given the equation of a quadratic graph, state, without plotting, the coordinates of the vertex, the equation of the axis of symmetry and state whether the parabola has a maximum or a minimum.

Type 2: Given the coordinates of the vertex and the coordinates of one other point through which the graph passes, pupils are to find the equation.

Self mark exercise 4

The following are equations of graphs of a parabola. In each case

(i) state whether the parabola has a maximum or a minimum value. Give the value of the maximum / minimum.

(ii) state the coordinates of the vertex.

(iii) state the equation of the line of symmetry (axis) of the parabola.

(iv) make a sketch.

1. $y = 2x^2 - 3$
2. $y = {}^{-}4(x - 3)^2$
3. $y = 5 - (x + 4)^2$
4. $y = 2(x + 3)^2 - 4$

In the following questions you are given the coordinates of the vertex V of a parabola and the coordinates of one point through which the parabola passes. Find the equation of the parabola.

5. $V({}^{-}2, 3)$ and passing through $({}^{-}3, 5)$
6. $V(3, {}^{-}2)$ and passing through $(5, {}^{-}10)$
7. $V({}^{-}3, {}^{-}4))$ and passing through $({}^{-}4, {}^{-}6)$

Check your answers at the end of this unit.

III Graphs with equations $y = ax^2 + bx + c$ $(a \neq 0)$ (I)

The graph of the quadratic relation $y = ax^2 + bx + c$ is called a parabola.

To plot the graph:

1. Construct a table of corresponding values of x and y. (at least six)
2. Plot the points with coordinates determined by the x and y values in your table on a labelled coordinate grid. (Indicate the position of the points with X)
3. Join the plotted points with a **smooth** curve. Do NOT join the points with straight line segments. (Why not?) Be aware that the vertex need not be one of the points in the table.
4. Label the graph specifically with its equation if more than one graph is drawn on the same coordinate axes system.

Self mark exercise 5

1. The height h in metres of a stone thrown into the air t seconds after being thrown is given by

   $h = {}^{-}5t^2 + 50t$

   a) Make a table of corresponding values of t (ranging from 0 to 10 seconds) and h (m).

   b) On a coordinate grid plot h against t. Use on the horizon t-axis 1 cm to represent 1 second. On the vertical h axis use 1 cm to represent 10 m. Remember to use X (intersection of two small lines) to indicate a point on your grid.

   c) The graph has an axis of symmetry. What is its equation?

   d) How can you see from the table in a) that the graph must be symmetrical?

   e) What is the maximum height reached by the stone?

2. a) Make a table to plot the graph of the parabola with equation $y = x^2 - 2x - 8$. Take ${}^{-}3 \leq x \leq 5$.

   b) What is the equation of the axis of the parabola?

   c) What are the coordinates of the vertex?

   d) Where does the parabola cross the x-axis ($y = 0$)?

   e) What quadratic equation has the x-values of d) as solution?

3. a) Make a table to plot the graph of the parabola with equation $y = {}^{-}x^2 + 5x$. Take ${}^{-}2 \leq x \leq 6$.

   b) What is the equation of the axis of the parabola?

   c) What are the coordinates of the vertex?

   d) Where does the parabola cross the x-axis ($y = 0$)?

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e) What quadratic equation has the x-values of d) as solution?

4. a) Make a table to plot the graph of the parabola with equation $y = {}^{-}x^2 + x + 6$.

   b) What is the equation of the axis of the parabola?

   c) What are the coordinates of the vertex?

   d) Where does the parabola cross the x-axis ($y = 0$)?

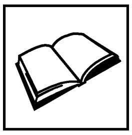
   e) What quadratic equation has the x-values of d) as solution?

Check your answers at the end of this unit.

IV Graphs with equation $y = ax^2 + bx + c$, $a \neq 0$ (II)

The equation $y = ax^2 + bx + c$ can be written in the form $y = a(x - p)^2 + q$ by the technique of completing the square. In the latter format coordinates of the vertex and the equation of the line of symmetry can be readily obtained.

Section B4:
Solving quadratic equations by graphical methods



The graphical method of solving quadratic equations introduces a general technique for solving any type of equation. As most equations cannot be solved by exact algebraic processes, a graphical approach is a valid alternative. Where possible, graphic calculators and computers should be used to illustrate the techniques.

The basic concept is linking the solution of simultaneous equations to the intersection of graphs. Solving the system

$$\begin{cases} f(x) = 0 \\ g(x) = 0 \end{cases}$$

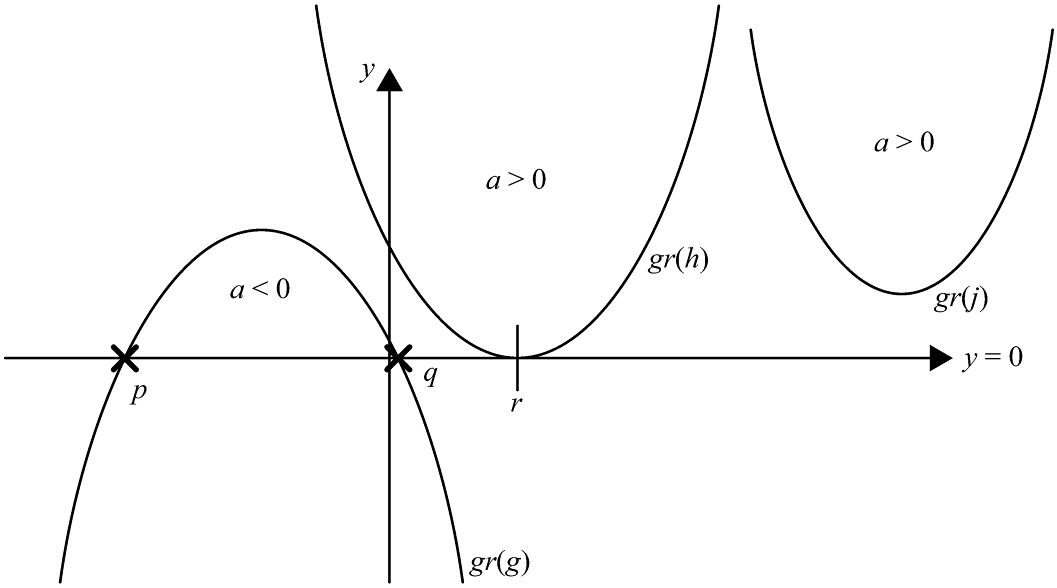
is interpreted as finding the x-coordinate of the points of intersection of the graphs with equation $y = f(x)$ and $y = g(x)$, i.e., looking for the x-values that make $f(x) = g(x)$.

In the case of the quadratic equation $ax^2 + bx + c = 0$ the two graphs involved are the graphs with the equation $y = ax^2 + bx + c$ and $y = 0$ (the x-axis).

The x-coordinate of the point(s) of intersection (if any) of $y = ax^2 + bx + c$ with the x-axis.

($y = 0$) are the solutions to the equation $ax^2 + bx + c = 0$.

Illustrating the number of solutions of $ax^2 + bx + c = 0$.



Graph $gr(g)$ illustrates 2 solutions of the equation $ax^2 + bx + c = 0$; $x = p$ or $x = q$ are the solutions. The graph intersects the x-axis in two distinct points.

For example, $x^2 + x - 6 = 0 \quad \Leftrightarrow (x + 3)(x - 2) = 0$ has the solution $x = {}^-3$ or $x = 2$.

The graph with equation y = $x^2 + x - 6$ cuts the x-axis at $({}^-3, 0)$ and $(2,0)$.

Graph $gr(h)$ is touching the x-axis and the equation of $ax^2 + bx + c = 0$ has one solution ($x = r$). In this case some still like to say that there are two solutions but that the two coincide with each other.

For example, $(x - 2)^2 = 0$ has as solution $x = 2$. The graph with equation $y = (x - 2)^2$ touches the x-axis at $(2, 0)$.

Graph $gr(j)$ has no points of intersection with the x-axis. The equation $ax^2 + bx + c = 0$ has no (real) solutions in this case.

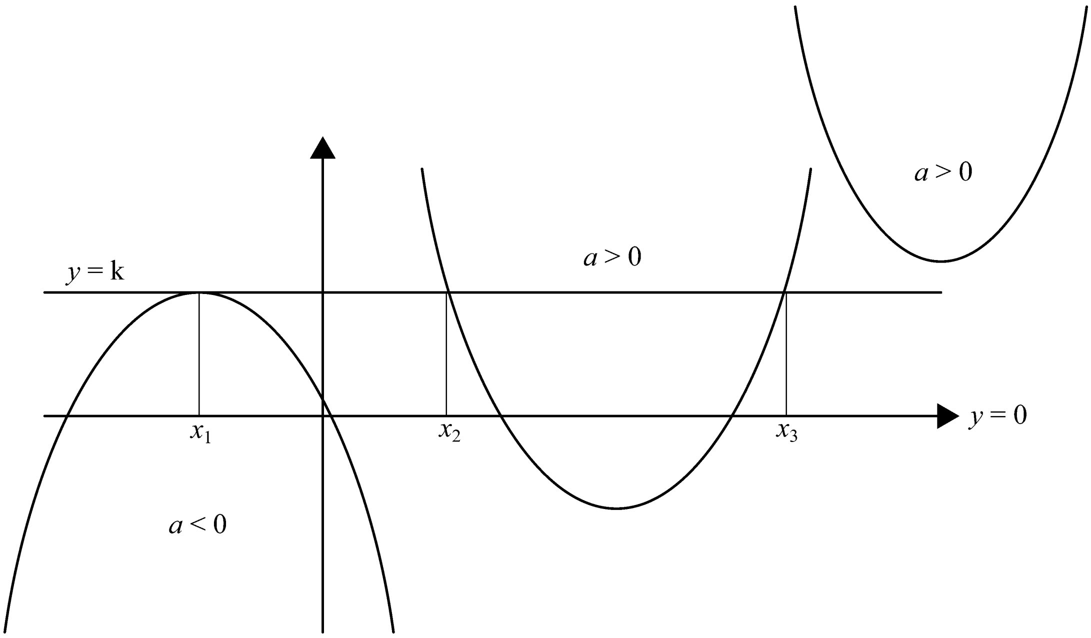
For example, the equation $x^2 + 2x + 4 = 0$ has no (real) solutions. The graph with equation

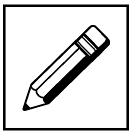
$y = x^2 + 2x + 4$ does not intersect the x-axis [the vertex is at $({}^-1,3)$].

In the case of the quadratic equation $ax^2 + bx + c = k$ the two graphs involved are the graphs with the equations $y = ax^2 + bx + c$ and $y = k$ (the line parallel to the x-axis).

The x-coordinate of the point(s) of intersection of $y = ax^2 + bx + c$ with the horizontal line with equation $y = k$ are the solutions to the equation $ax^2 + bx + c = k$.

Illustrating the number of solutions of $ax^2 + bx + c = k$.





Self mark exercise 6

1. a) Make a table to plot the graph of the parabola with equation $y = x^2 - 4x - 5$.

   Take $^-2 \le x \le 6$.

   Label your x-axis from $^-3$ to 7 and the y-axis from $^-10$ to 10.

   b) Draw the line of symmetry. What is its equation? What are the coordinates of the vertex?

   c) Use your graph to find the values of x if $y = 0$ (the zeros or x-intercepts).

   d) The x-values in b) are the solutions of a quadric equation. Write down that equation.

   e) Consider the equation $x^2 - 4x - 5 = 7$ and compare with $y = x^2 - 4x - 5$.

   What value of y gives the equation $x^2 - 4x - 5 = 7$?

   How can you now use your graph to solve the equation $x^2 - 4x - 5 = 7$?

   Find the solutions to this equation.

   f) Using your graph solve the equation $x^2 - 4x - 5 = {}^-5$.

   g) Using your graph solve the equation $x^2 - 4x - 5 = {}^-9$.

   h) Using your graph solve the equation $x^2 - 4x - 5 = {}^-10$.

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i) Solve each of the equations $x^2 - 4x - 5 = 0$, $x^2 - 4x - 5 = 7$, $x^2 - 4x - 5 = {}^-5$, $x^2 - 4x - 5 = {}^-9$ and $x^2 - 4x - 5 = {}^-10$ using the factorisation method.

j) Which method do you feel is better to solve a quadratic equation? Justify your choice.

2. a) Make table to plot the graph of the parabola with equation $y = x^2 + 4x - 4$.

Take ${}^-6 \le x \le 2$.

Label your x-axis from ${}^-7$ to 3 and the y-axis from ${}^-10$ to 10.

b) Draw the line of symmetry. What is its equation? What are the coordinates of the vertex?

c) Use your graph to find the values of x if $y = 0$ (the zeros or x-intercepts). Can you find them exactly?

d) The x-values in b are the solutions of a quadric equation. Write down that equation.

e) Consider the equation $x^2 + 4x - 4 = 8$ and compare with $y = x^2 + 4x - 4$.

What value of y gives the equation $x^2 + 4x - 4 = 8$?

How can you now use your graph to solve the equation $x^2 + 4x - 4 = 8$?

Find the solutions to this equation.

f) Using your graph solve the equation $x^2 + 4x - 4 = {}^-4$.

g) Using your graph solve the equation $x^2 + 4x - 4 = {}^-8$.

h) Using your graph solve the equation $x^2 + 4x - 4 = {}^-10$.

i) Using your graph solve the equation $x^2 + 4x - 4 = 4$.

j) Using your graph solve the equation $x^2 + 4x - 4 = {}^-2$.

k) Try to solve each of the equations $x^2 + 4x - 4 = 0$, $x^2 + 4x - 4 = 8$, $x^2 + 4x - 4 = {}^-4$,

$x^2 + 4x - 4 = {}^-8$, $x^2 + 4x - 4 = 4$ and $x^2 + 4x - 4 = {}^-2$ using the factorisation method.

l) Does the factorisation method always work? When does it work? When does it fail?

m) How many solutions can the equation $x^2 + 4x - 4 = a$ have? For which values of a?

Check your answers at the end of this unit.

The above self mark exercise can be given, in adapted form if necessary, to pupils as a discussion exercise. It should clarify to pupils the basic ideas involved in solving quadratic equations using graphs. Consolidation (and extension) will be needed.

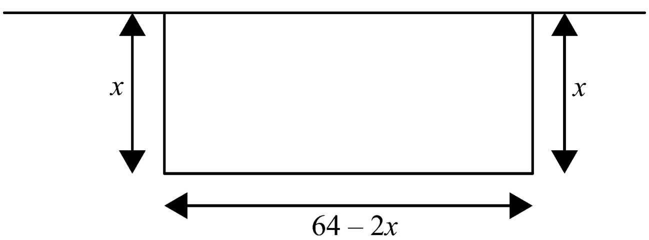
Here are some suggested questions for applications in context:

1. A ball is thrown upwards from the top of a tall building. The height h metres above the ground, t seconds after the ball was thrown up is given by $h = {}^{-}5t^2 + 10t + 75$.
   a) Construct a table of values of h and t and plot the graph.
   b) What is the height of the building?
   c) What is the maximum height above the ground reached by the ball?
   d) After how many seconds is the ball again at the level of the top of the building?
   e) After how many seconds will the ball hit the ground?
2. A ball is kicked by a player. The ball follows a parabolic path given by $h = {}^{-}\frac{1}{10}a^2 + 2a$.

   a represents the distance from the player kicking the ball and h the height of the ball, both in metres.
   a) Construct a table and plot the path followed by the ball.
   b) After how many metres does the ball hit the ground again?
   c) What is the maximum height reached by the ball?
   d) The player was trying to reach another player 18 m away from him. Will the ball reach the other player? Explain.

Self mark exercise 7

1. Solve the application questions 1 - 2 listed above.
2. A rectangular plot, one side bordering a river, is to be fenced. 64 m of fencing is available. What is the maximum possible area that can be fenced? Plot a graph.



Check your answers at the end of this unit.

The use of graphs in solving quadratic equations allows for flexibility as the equation to be solved can always be transformed into equivalent format.

For example:

To solve $x^2 - 2x - 8 = {}^{-}5$, you could:

(i) plot the graphs of $y = x^2 - 2x - 8$ and $y = {}^{-}5$ and read the

x-coordinate(s) of point(s) of intersection.

(ii) Rewrite $x^2 - 2x - 8 = {}^{-}5$ as $x^2 = 2x + 3$ and plot the graphs of

$y = x^2$ and $y = 2x + 3$ and read the x-coordinate(s) of point(s) of intersection.

Self mark exercise 8

1. Solve graphically the equation $x^2 - 2x - 8 = {}^{-}5$ using the two methods suggested above.

   Which one do you prefer? Justify your answer.

2. Solve graphically the equation $x^2 - 3x - 7 = {}^{-}8$ using two different graphical methods. Compare and comment.

Check your answers at the end of this unit.

Section B5: Solving quadratic equations by completing the square

Equations of the format $x^2 = p$ and $(x + a)^2 = p$ do solve rather straight forward.

For $p < 0$ there is no (real) solution, for $p = 0$ you have one solution and for p > 0 you have two distinct solutions $x = \sqrt{p}$ or $x = {}^{-}\sqrt{p}$ for the first equation and $x = {}^{-}a + \sqrt{p}$ or $x = {}^{-}a - \sqrt{p}$ for the second equation.

Hence if you would express any quadratic equation of the form $ax^2 + bx + c = 0$ in the form $a(x + p)^2 + q = 0$ which can be restructured to the form $(x + p)^2 = \frac{{}^{-}q}{a}$, the equation will solve ‘easily’.

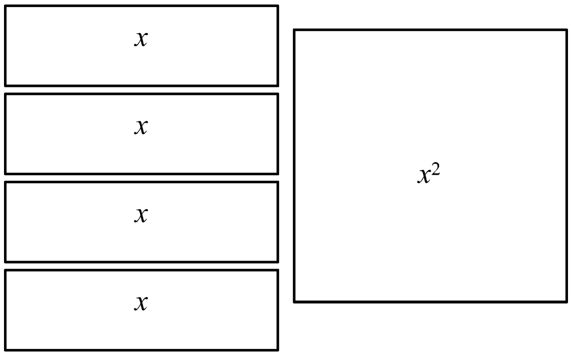
Expressing $ax^2 + bx + c$ in the form $a(x + p)^2 + q$ is called completing the square.

Why completing the square? Using your algebra tiles will make this clear.

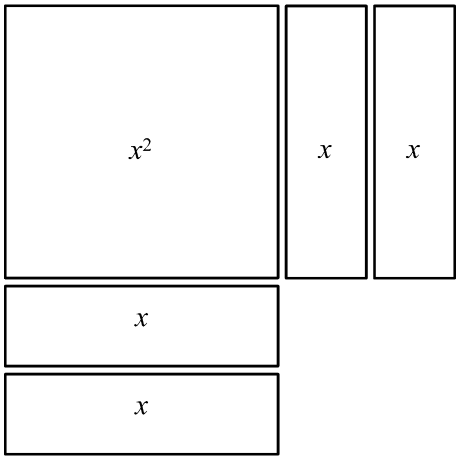
Example:

Complete the square of $x^2 + 4x - 1$.

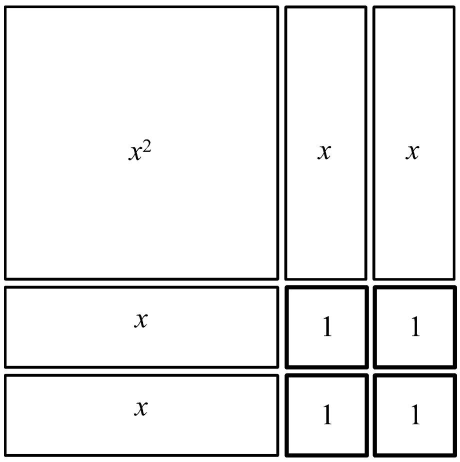
Take one tile to represent x^2 and 4 tiles representing x. Note you do not need the number value ($^-1$).



Place these tiles such as to form a square.



You can make a square with side $(x + 2)$ but you are 4 unit tiles short to complete the square. If you add them you have completed the square.



In order to keep an equality you will have to subtract the added 4 tiles.

So you have now $x^2 + 4x = (x + 2)^2 - 4$.

Or the original expressions become

$x^2 + 4x - 1 = (x + 2)^2 - 4 - 1$

$x^2 + 4x - 1 = (x + 2)^2 - 5$

Solving $x^2 + 4x - 1 = 0$ is the same as solving $(x + 2)^2 - 5 = 0$.

The form gives $x^2 + 4x - 1 = 0$ problems because our method of factorisation does not work. The completed square form (which is equivalent to the first equation) solves ‘easily’ as follows:

$(x + 2)^2 - 5 = 0$

$(x + 2)^2 = 5$

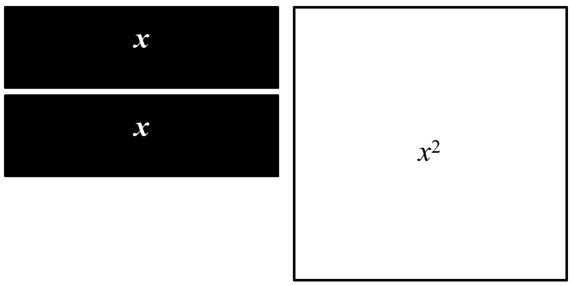
$x + 2 = \sqrt{5}$ or $x + 2 = {}^{-}\sqrt{5}$

$x = {}^{-}2 + \sqrt{5}$ or $x = {}^{-}2 - \sqrt{5}$

Let’s look at another example: solving by completing the square $x^2 - 2x - 4 = 0$.

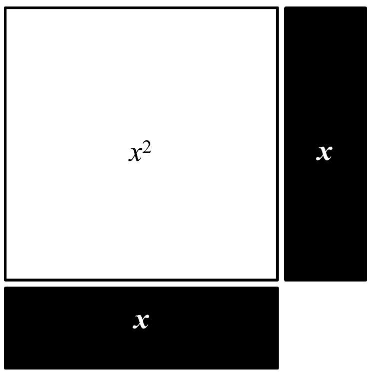
Write $x^2 - 2x = 4$ and try to write the left hand side as a complete square.

Using tiles you have:



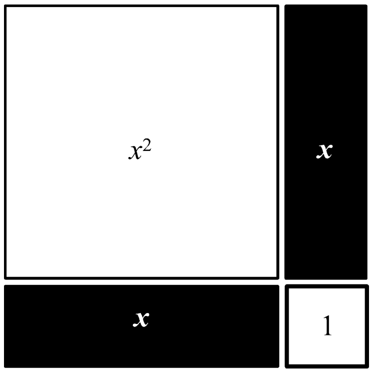
The shaded tiles represent negatives.

Place these tiles such as to form a square.



For easy representation the negatives have been placed next to the square—so we can clearly see by what we are short—instead of on top (as you have been doing when using tiles to factorise expressions).

You can make a square with side $(x - 1)$ but you are 1 unit tile short to complete the square. If you add that tile you have completed the square.



You have now $\quad x^2 - 2x = 4$

$$(x - 1)^2 - 1 = 4$$

The left hand side is the completed square.

This solves as $(x - 1)^2 = 5 \quad x = 1 + \sqrt{5}$ or $x = 1 - \sqrt{5}$

Let's tabulate what you have got now:

$x^2 + 4x = (x + 2)^2 - 4$

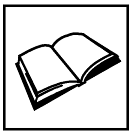
$x^2 - 2x = (x - 1)^2 - 1$

Self mark exercise 9

Complete the blanks in the following cases:

1. $x^2 - 6x = (x - 3)^2 -$ ___
2. $x^2 + 8x = (x +$ __ $)^2 -$ ___
3. $x^2 - 4x = (x -$ __ $)^2 -$ ___
4. $x^2 + 5x = (x + 2.5)^2 -$ ___
5. $x^2 - 3x = (x -$ __ $)^2 -$ ___
6. $x^2 + x = (x +$ __ $)^2 -$ ___
7. $x^2 - 7x = (x -$ __ $)^2 -$ ___
8. $x^2 + 2ax = (x +$ __ $)^2 -$ ___
9. $x^2 - 2bx = (x -$ __ $)^2 -$ ___
10. $x^2 + 10x =$ ___
11. $x^2 - 11x =$ ___
12. $x^2 + \frac{1}{2}x =$ ___
13. $x^2 - \frac{1}{3}x =$ ___
14. $x^2 + bx =$ ___

Check your answers at the end of this unit.



What if the coefficient of x^2 is not 1 as in all the examples?

This is a minor problem: we can always place the coefficient of x^2 as a factor outside brackets and complete the square inside the brackets as usual.

Here is an example:

$3x^2 + 6x = 3(x^2 + 2x) = 3\{(x + 1)^2 - 1\} = 3(x + 1)^2 - 3$

Solving, by completing the square, $3x^2 + 6x - 4 = 0$ goes like this:

$$3x^2 + 6x = 4$$

$$3(x^2 + 2x) = 4$$

$$3\{(x + 1)^2 - 1\} = 4$$

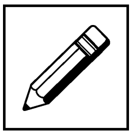
$$3(x + 1)^2 - 3 = 4$$

$$3(x + 1)^2 = 7$$

$$(x + 1)^2 = \frac{7}{3}$$

$$x + 1 = \sqrt{\frac{7}{3}} \qquad \textit{or} \qquad x + 1 = {}^-\sqrt{\frac{7}{3}}$$

$$x = {}^-1 + \sqrt{\frac{7}{3}} \qquad \textit{or} \qquad x = {}^-1 - \sqrt{\frac{7}{3}}$$



Self mark exercise 10

Solve the following equations by completing the square

1. $x^2 + 3x + 1 = 0$ 2. ${}^-x^2 - 4x + 6 = 0$ 3. $2x^2 + 2x - 1 = 0$
4. ${}^-5x^2 + 3x + 2 = 0$ 5. $4x^2 + 6x - 3 = 0$ 6. $ax^2 + bx + c = 0$

Check your answers at the end of this unit.

Section B6:
Solving quadratic equations by using the formula

If you apply the completing of the square technique to the general quadratic equations it leads to an expression for the roots generally known as the abc - formula or quadratic formula.

If you succeeded in the previous self mark exercise question 6, you have already derived the quadratic formula.

If you found it difficult—what about your pupils? Here are the steps:

Deriving the quadratic formula.

$ax^2 + bx + c = 0$

$ax^2 + bx = - c$

$a(x^2 + \frac{b}{a}x) = {}^-c$ taking *a* outside the bracket

$a\left\{\left(x+\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right\} = {}^-c$ completing the square

$\left(x+\frac{b}{2a}\right)^2 - (\frac{b^2}{4a^2} = {}^-\frac{c}{a}$ dividing by $a \neq 0$

$\left(x+\frac{b}{2a}\right)^2 = (\frac{b^2}{4a^2} - \frac{c}{a}$ rearranging

$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$ fractions at RHS under one denominator

$x + \frac{b}{2a} = \sqrt{\frac{b^2-4ac}{4a^2}}$ or $x + \frac{b}{2a} = {}^-\sqrt{\frac{b^2-4ac}{4a^2}}$

$x = {}^-\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}$ or $x = {}^-\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}$

$x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ or $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$

Self mark exercise 11

1. Solve the quadratic equations using the formula, giving your answers to 2 decimal places:

   a) $3.2x^2 - 4.1x - 7.9 = 0$

   b) $6.7x^2 + 5.3x - 4.7 = 0$

2. Which part of the quadratic formula determines whether the quadratic equation has two, one or no solutions? Explain and illustrate with examples and graphs.

Check your answers at the end of this unit.

Section B7: Solving quadratic equations by using trial and improvement techniques

The trial and improvement technique is a general method for solving equations. It is important to know how many roots the equation has—sketch graphs might be of help here—as otherwise one might forget some of the roots. The trial and improvement method therefore often goes together with making a sketch. The sketch will help to find a first reasonable guess for the root(s). These can be improved to the required accuracy. As already stated when looking at this method when discussing methods to solve linear equations: the technique—although very general—is not the most appropriate one for linear / quadratic equations as for these other techniques are available. In cases where no algebraic solution of an equation can be found the trial and improvement technique is appropriate. Graphic calculators and computers (spreadsheet programme) can assist to avoid unpleasant computational work.

For the quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$) the number of roots is determined by the value of the **discriminant** $D = b^2 - 4ac$, the expression under the root symbol in the quadratic formula. As roots from negative numbers are not real, the value of D has to be non negative, i.e., $D \geq 0$.

$D < 0$ no (real) solutions of the equation.

$D = 0$ one solution (or two coinciding solutions).

$D > 0$ two distinct solutions.

Example:

Solve using trial and improvement method the equation $3x^2 + 5x + 1 = 0$

$D = 5^2 - 4 \times 3 \times 1 = 13$. Hence you are to look for two roots.

Where are these root(s) approximately?

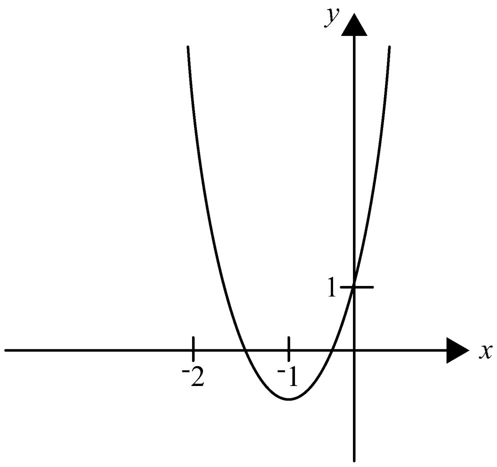
Take some values of x with the corresponding value of $y = 3x^2 + 5x + 1$

$x = 0$ $\quad y = 1$

$x = ^-1$ $\quad y = ^-1$ $\quad$ so there must be a root between $x = 0$ and $x = ^-1$ (start with trying $x = ^-0.5$)

$x = ^-2$ $\quad y = 3$ $\quad$ so the other root is between $^-1$ and $^-2$ (start with trying $^-1.5$)

The sketch graph illustrates the graph with equation $y = 3x^2 + 5x + 1$



Approximating the root in the interval ($^-1$, 0)

x	y	too small/big?
0	1	too big
$^-0.5$	$^-0.75$	too small
$^-0.25$	$^-0.0625$	too small
$^-0.12$	0.472	too big
$^-0.23$	0.0087	too big
$^-0.235$	$^-0.0093$	too small
$^-0.232$	0.00147	too big

To two decimal places $x = ^-0.23$

Self mark exercise 12

1. Use trial and improvement to obtain the other root of the equation to 2 dp.

Check your answers at the end of this unit.

Unit 3, Practice task

1. Write 5 questions / problem situations leading to quadratic equations to motivate pupils to learn about techniques for solving quadratic equations.

   Develop a work sheet with these questions and try it out in your class prior to covering the techniques of solving quadratics.

   Write an evaluative report.

2. Compare each of the techniques for solving quadratic equations:
   factorisation
   graphical solution
   completing the square
   formula
   trial and improvement

   Which of these methods do you consider appropriate for pupils at lower secondary school? Justify your answer.

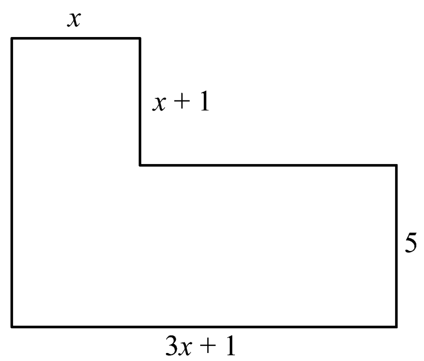
3. Which of the above methods do you consider most appropriate to solve each of the following questions? Justify your choice.

   a) The height of a triangle is x cm and the corresponding base is $(x + 8)$ cm long. If the area enclosed by the triangle is 42 cm^2 , find the height and base of the triangle.

   b) A rectangle encloses an area of 55 cm^2. Its sides are of length x cm and $(x + 6)$ cm. Find the length of the sides.

   c) The product of a number and three less than the number is 50. What is the number?

   d) The area enclosed by the shape below is 90 cm^2. Find the length of each of the sides of the shape.



e) The product of two consecutive even numbers is 440. Find the numbers.

f) A man walks due West and then North the same distance plus 4 km. He is now 20 km from his starting point. How far West did he walk first?

g) A right-angled triangle has right-angled sides of x cm and $(21 - x)$ cm. If the hypotenuse is 15 cm long, find the length of the right-angled sides.

h) The length of a rectangle exceeds double its width by 4 cm. The diagonal is 26 cm, Find the length of the rectangle.

i) The area of a rectangle with sides $(x - 3)$ m and $(x - 6)$ m is 28 m^2. Find the length of the sides of the rectangle.

4. Would you derive the quadratic formula with your pupils? Justify your choice.

   If you answered NO: would you introduce your pupils to the formula at all? If yes: how would you introduce the formula if you do not derive it? Would that not be teaching by intimidation? (Telling without justification. Presenting the formula as an article of faith.)

Present your assignment to your supervisor or study group for discussion.

Summary

This unit on quadratic equations has continued the structured approach of the previous unit on linear equations. That is, the standard manipulations have been contextualized via realistic examples and manipulatives and, for this topic, extensive visualizations using graphs. The unit indicated that not all its solving methods are appropriate for this age group. Indeed, in this calculator age, the only method that will remain useful to your students is the quadratic equation and its graph. Be judicious when you decide how much quadratic knowledge you expect your students to retain.

Unit 3: Answers to self mark exercises

Self mark exercise 1

Use numerical examples with both positive and negative values for a.

Self mark exercise 2

1. a) $R^2 = L^2 + (R - x)^2$

   b) $10x(x + 8) = 1000$ $12w(w + 10) = 1000$

Self mark exercise 3

1) $^{-}2$, 12	2) 4, 6	3) $^{-}12$, 2
4) 2, 12	5) $^{-}3$, 8	6) $^{-}1$, 24
7) $^{-}5$, 5	8) $h = 6$ cm, $b = 4$ cm or $h = 8$ cm, $b = 3$ cm	
9) 4 cm × 5 cm	10) 9	11) 5 cm
12) $x = 3$	13) 21, 23 or $^{-}21$, $^{-}23$	
14) 8 km	15) a/b (i) $p > 0$ (ii) $p = 0$ (iii) $p < 0$	
16) False	17) False	

18) (i) Ask pupil to explain (ii) Create mental conflict by checking the given answers in the original equation (iii) Resolve the conflict in discussion with the pupil. Establish first the basic principle:
If $AB = 0 \Rightarrow A = 0$ or $B = 0$ and $AB = C$ (C not zero) does not imply $A = C$ or $B = C$. Hence quadratic equations need a 0 at one side of the equal sign when solving (insist on this initially even in cases such as $x^2 = 4$).
Next try to factorise and apply the basic principle. (iv) Give consolidation exercise with quadratic equations that can be solved using factorisation.

Self mark exercise 4

1. Minimum $^{-}3$, vertex $(0, {}^{-}3)$, axis $x = 0$
2. Maximum 0, vertex (3, 0), axis $x = 3$
3. Maximum 5, vertex $({}^{-}4, 5)$, axis $x = {}^{-}4$
4. Minimum $^{-}4$, vertex $({}^{-}3, {}^{-}4)$, axis $x = {}^{-}3$
5. $2(x + 2)^2 + 3$
6. $^{-}2(x - 3)^2 - 2$
7. $^{-}2(x + 3)^2 - 4$

Self mark exercise 5

1. a)

t	0	1	2	3	4	5	6	7	8	9	10
h	0	45	80	105	120	125	120	105	80	45	0

c) $t = 5$ d) symmetry in table about $t = 5$ e) 125 m

2. a)

x	$^-3$	$^-2$	$^-1$	0	1	2	3	4	5
y	7	0	$^-5$	$^-8$	$^-9$	$^-8$	$^-5$	0	7

b) $x = 1$ c) V(1, $^-9$) d) ($^-2$, 0), (4, 0) e) $x^2 - 2x - 8 = 0$

3. a)

x	$^-2$	$^-1$	0	1	2	3	4	5	6
y	$^-14$	$^-6$	0	4	6	6	4	0	$^-6$

b) $x = 2.5$ c) V(2.5, 6.25) d) (0, 0), (5, 0) e) $^-x^2 + 5x = 0$

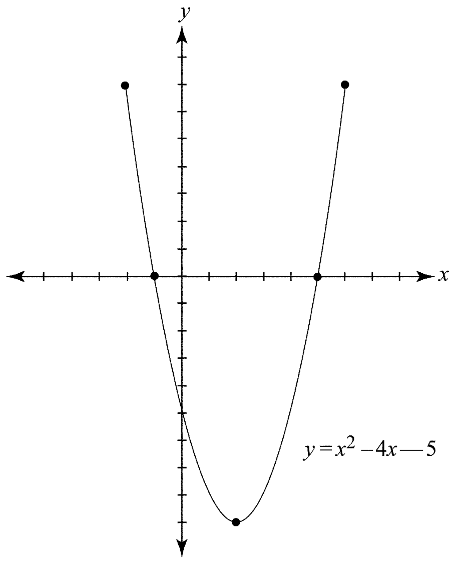
4. a)

x	$^-3$	$^-2$	$^-1$	0	1	2	3	4
y	$^-6$	0	4	6	6	4	0	-6

b) $x = 0.5$ c) V(0.5, 6.25) d) ($^-2$, 0), (3, 0) e) $^-x^2 + x + 6 = 0$

Self mark exercise 6

1. a)

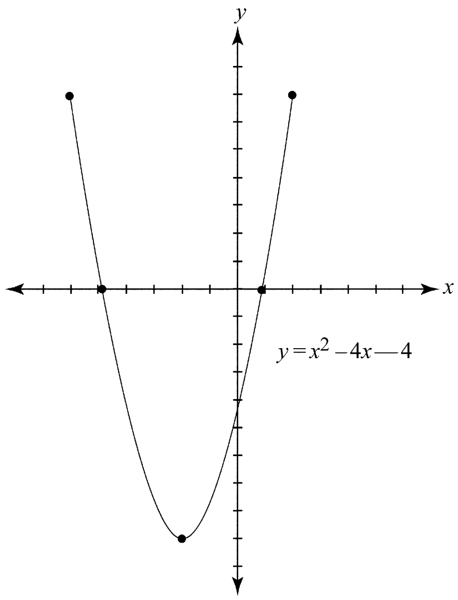


b) $x = 2$, V(2, $^-9$) c) $x = {}^-1$, $x = 5$ d) $x^2 - 4x - 5 = 0$

e) intersect graphs with equations $y = x^2 - 4x - 5$ and $y = 7$; $x = 6$, $x = {}^-2$

f) $x = 0$, $x = 4$ g) $x = 2$ h) No (real) solutions

2. a)



b) $x = {}^-2$, $V({}^-2, {}^-8)$ c) $x = {}^-4.8$, $x = 0.8$ (1 dp)

d) $x^2 + 4x - 4 = 0$

e) intersect graphs with equations $y = x^2 + 4x - 4$ and
$y = 8$; $x = {}^-6$, $x = 2$

f) $x = 0$, $x = {}^-4$ g) $x = {}^-2$ h) No (real) solutions

i) $x = {}^-5.5$, $x = 1.5$ (1dp) j) $x = {}^-4.5$, $x = 0.5$ (1dp)

l) Factorisation is over the integers.

m) One solution for a = $^-8$, two solutions for a > $^-8$,
no solution for a < $^-8$.

Self mark exercise 7

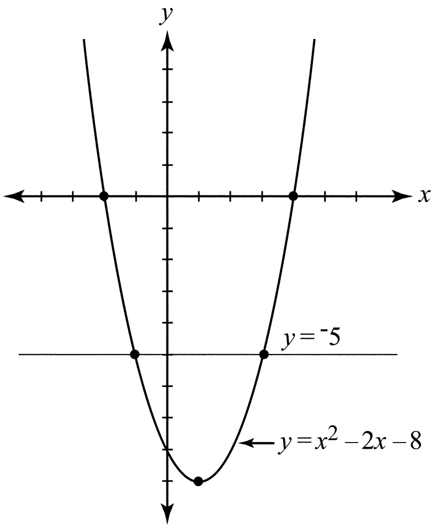
1. Solution to **1b** 75 m **1c** 80 m **1d** 2 s **1e** 5 s

   Solution to **2b** 20 m **2c** 10 m **2d** No, at position of player, ball is still 3.6 m above ground level and will pass over the head of the player.

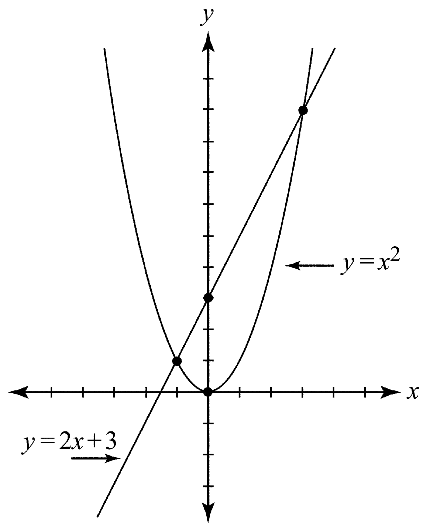
2. 512 m^2

Self mark exercise 8

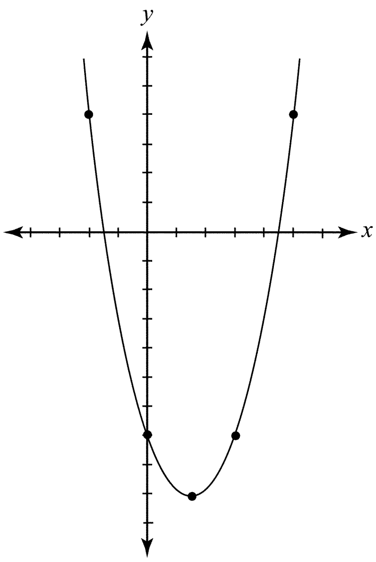
1. (i)



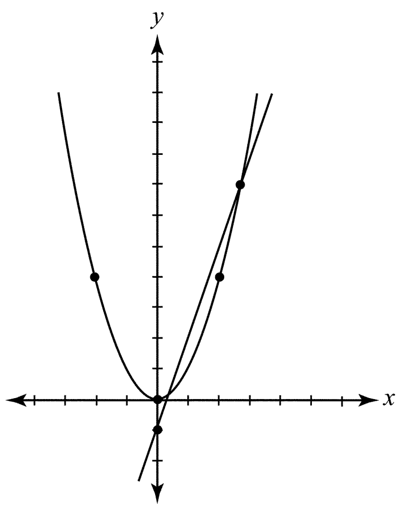
(ii)



2. (i)



(ii)



Self mark exercise 9

1. $x^2 - 6x = (x - 3)^2 - 9$
2. $x^2 + 8x = (x + 4)^2 - 16$
3. $x^2 - 4x = (x - 2)^2 - 4$
4. $x^2 + 5x = (x + 2.5)^2 - 6.25$
5. $x^2 - 3x = (x - 1.5)^2 - 2.25$
6. $x^2 + x = (x + 0.5)^2 - 0.25$
7. $x^2 - 7x = (x - 3.5)^2 - 12.25$.
8. $x^2 + 2ax = (x + a)^2 - a^2$
9. $x^2 - 2bx = (x - b)^2 - b^2$
10. $x^2 + 10x = (x + 5)^2 - 25$
11. $x^2 - 11x = (x - 5.5)^2 - 30.25$
12. $x^2 + \frac{1}{2}x = \left(x + \frac{1}{4}\right)^2 - \frac{1}{16})$
13. $x^2 - \frac{1}{3}x = \left(x - \frac{1}{6}\right)^2 - \frac{1}{36}$
14. $x^2 + bx = \left(x + \frac{1}{2}b\right)^2 - \frac{1}{4}b^2$

Self mark exercise 10

1. $x = {}^{-}1\frac{1}{2} + \frac{1}{2}\sqrt{5}$, $x = {}^{-}1\frac{1}{2} - \frac{1}{2}\sqrt{5}$
2. $x = {}^{-}2 + \sqrt{10}$, $x = -2 - \sqrt{10}$
3. $x = {}^{-}\frac{1}{2} + \frac{1}{2}\sqrt{3}$, $x = {}^{-}\frac{1}{2} - \frac{1}{2}\sqrt{3}$
4. $x = 1$, $x = \frac{2}{5}$
5. $x = \frac{1}{4}$, $x = {}^{-}1\frac{3}{4}$
6. $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$, $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

Self mark exercise 11

1. a) 2.34, $^{-}1.06$ b) 0.53, $^{-}1.32$
2. $b^2 - 4ac$ (= D) determines the number of roots

Self mark exercise 12

1. $x = {}^{-}1.43$ (2 dp)

Unit 4: Simultaneous equations

Introduction to Unit 4

In the previous unit you considered equations containing one variable. Real life situations are most of the time more complex: several variables are involved and the variables might be related in more than one way. The simplest case is to have a situation with two variables, related by two different equations. Such a system is referred to as a system of two linear equations in two variables.

Purpose of Unit 4

In this unit you will look at simultaneous equations in two variables and some techniques to solve them. You will mainly look at systems of two simultaneous equations in which both equations are linear. But even at the lower secondary school level it is not always necessary to restrict to systems of linear simultaneous equations, as the techniques are more generally applicable. You will start by looking at situations leading to a system of two (linear) equations—to create a need for solving techniques (section A1). You will meet three techniques for solving a system of two (linear) simultaneous equations

Section A2 Solving by graphical method

Section A3 Solving by substitution method

Section A4 Solving by elimination method

You will find other techniques, for example, using matrices, not covered here or in most mathematics books.

Objectives

When you have completed this unit you should be able to:

- present realistic situations to your pupils to create a need for techniques of solving systems of two simultaneous equations.
- solve systems of two simultaneous equations by (i) graphical method (ii) substitution method (iii) elimination method.
- justify which of the three techniques is most appropriate in a given situation.

Time

To study this unit will take you about 10 hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you are planned to cover the topic.

Unit 4: Simultaneous equations

Section A1: Situations leading to simultaneous equations

If you have 150 m of fencing material and you want to fence a rectangular plot with an area of 1400 m^2, how are you going to take the length and width of your rectangular plot?

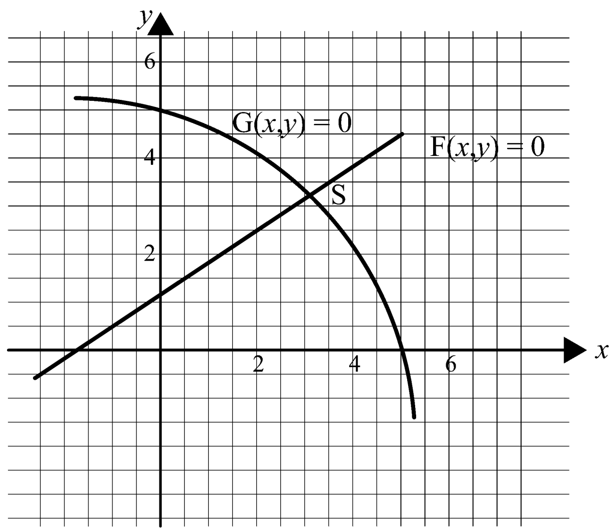
If you take the length of your plot to be *L* m and the width to be *W* m, you know that

$2(L + W) = 150$ and that $LW = 1400$. You have at the same time (simultaneously) two bits of information related by the two variables *L* and *W*.

In mathematics you would say: I have formed a system of 2 simultaneous equations.

In general if you have two variables and the two variables are related by two equations, you have a system of two simultaneous equations. The notation used is

$$\begin{cases} F(x,y)=0 \\ G(x,y)=0 \end{cases}$$



There are numerous situations leading to systems of simultaneous equations. Here are some examples:

1. A school trip is organised. There are 45 places available on the transport. 6 are for teachers and the number of girls on the trip (*g*) will be 9 more than the number of boys (*b*). How many boys and how many girls will go on the trip?
2. I have P 75 to spend. I can buy 2 CDs and 3 empty cassettes with the money. I can also buy 1 CD and 9 empty cassettes with my money. If the cost of a CD is P *c* and the cost of an empty cassette is P *d*, find the values of *c* and *d*.

3. You want to make a closed box from cardboard. You have 1760 cm^2 of cardboard and want a closed box with a volume of 4800 cm^3. The box is to have a square base. What will be the measurements of the side of the square base and the height of the box?

Unit 4: Practice task

1. Form the system of simultaneous equation that should be solved to answer the question in the examples 1- 3 above.
2. In mathematics books frequently situations are presented, in so called word problems, to give a 'realistic' flavour, while in fact the situation is not realistic at all in the sense that one will never meet such a situation in real life. The questions are not practical but 'phoney'. Which of the above examples do you consider practical / realistic? Which do you consider 'phoney', i.e., constructed for the sake of creating a system? Justify your answer.
3. Are all situations (word problems) covered in mathematics practical, related to the world of work, realistic? Is there a place for 'phoney' questions? Support your view, illustrating with examples.
4. Make a worksheet with 10 situations for pupils to form simultaneous equations in order to motivate them to engage in activities covering the techniques of solving the systems.

Present your assignment to your supervisor or study group for discussion.

Section A2: Solving simultaneous linear equations by straight line graphs

Graphical solution of equations was covered earlier in this module. The technique discussed there applies equally to systems of simultaneous equations. In terms of graphs to solve the system

$$\begin{cases} F(x,y)=0 \\ G(x,y)=0 \end{cases}$$

You plot the graph of $F(x, y) = 0$ and of $G(x, y) = 0$. The coordinates of the point(s) of intersection of the two graphs will give you the (approximate) solution(s) (x, y) of the system of simultaneous equations, as the coordinates of the points of intersection satisfy the two equations simultaneously because they are on both graphs.

Example

Solve graphically

$$\begin{cases} 3x+2y=12 \\ 3x-2y=0 \end{cases}$$

a) Plot the graph of the line with equation $3x + 2y = 12$.

Remember: find the coordinates of three points—the x-intercept, the y-intercept and any other point.

If $x = 0$: $2y = 12$, $y = 6$ (the y-intercept)

If $y = 0$: $3x = 12$, $x = 4$ (the x-intercept)

If $x = 2$: $6 + 2y = 12 \quad 2y = 6 \quad y = 3$

Tabulated:

x	0	4	2
y	6	0	3

b) Plotting $3x – 2y = 0$

If $x = 0$, $y = 0$ (both x and y-intercepts are 0)

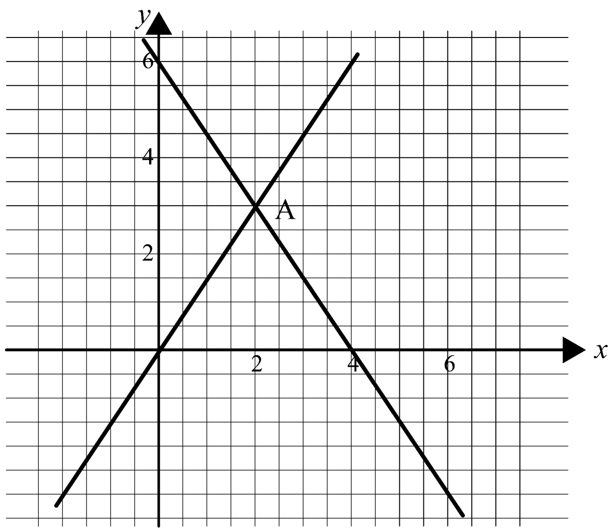
If $x = 4 \quad 12 – 2y = 0 \quad 2y = 12 \quad y = 6$

If $x = 2 \quad 6 – 2y = 0 \quad 2y = 6 \quad y = 3$

Coordinates are placed in the table:

x	0	4	2
y	0	6	3

From your graph read as accurately as possible the coordinates of the point of intersection.



At (2, 3) where the two lines intersect both equations are satisfied and hence the solution of the system is (2, 3).

Check the solution.

Self mark exercise 1

Solve graphically the simultaneous system of equations. Do not forget to check your values for x and y by substituting back in the original equations.

1. $\begin{cases} 4x + y = 14 \\ x + 5y = 13 \end{cases}$ 2. $\begin{cases} 2x - y = 5 \\ 3x - 2y = 4 \end{cases}$ 3. $\begin{cases} x - 2y = 4 \\ 5x - 6y = 18 \end{cases}$

4. $\begin{cases} 10x + 3y = 25 \\ 10x - 6y = 7 \end{cases}$ 5. $\begin{cases} xy = 1400 \\ x + y = 75 \end{cases}$

Check your answers at the end of this unit.

Section A3: Solving simultaneous linear equations by substitution

Some pupils might feel rightly that the graphical method used to solve simultaneous linear equations meets their needs. Especially when a graphic calculator / computer can be used, the zooming facility allows them to obtain the coordinates of the point of intersection to the accuracy required. In addition the graphical method applies to any two simultaneous equations and is not restricted to linear equations.

However in the absence of modern technology pupils can be asked to solve the system

$$\begin{cases} 10x + 3y = 25 \\ 10x - 6y = 7 \end{cases}$$

by plotting the lines with equation $10x + 3y = 25$ & $10x – 6y = 7$ and finding the coordinates of the point of intersection. (as you did in the previous exercise question 4).

Most likely you came up with (2, 2). However on checking this in the given equations it is found NOT to be the (exact) solution. This might motivate pupils to develop a method to find the exact solution ($x = 1.9$ & $y = 2$).

The solution of a simultaneous system by substitution works as follows.

(i) You make x or y the subject in one of the two given equations.

(ii) Substitute the expression for x (or y) obtained in (i) in the other equation.

This will give an equation in one variable only.

Example

A slot machine takes 50t and P1 coins only. There are 65 coins in the machine with a total value of P53.50. How many coins of each type are in the machine? (A 50t coin is worth half as much as a P1 coin.)

If the number of 50t is x and the number of P1 coins is y then $x + y = 65$ and $0.5x + y = 53.50$. The system to be solved is:

$$\begin{cases} x + y = 65 \\ 0.5x + y = 53.5 \end{cases}$$

$$\begin{cases} y = 65 - x \\ 0.5x + y = 53.5 \end{cases}$$ making y the subject in the first equation

$$\begin{cases} y = 65 - x \\ 0.5x + 65 - x = 53.5 \end{cases}$$ substituting y in the second equation

$$\begin{cases} y = 65 - x \\ {}^{-}0.5x + 65 = 53.5 \end{cases}$$ solving the second equation

$$\begin{cases} y = 65 - x \\ {}^{-}0.5x + 65 - 65 = 53.5 - 65 \end{cases}$$ solving the second equation

$$\begin{cases} y = 65 - x \\ {}^{-}0.5x = {}^{-}11.5 \end{cases}$$ solving the second equation

$$\begin{cases} y = 65 - x \\ x = 23 \end{cases}$$ solving the second equation ($^{-}11.5 \div {}^{-}0.5 = 23$)

$$\begin{cases} y = 65 - 23 \\ x = 23 \end{cases}$$ substituting the x-value in the first equation

$$\begin{cases} y = 42 \\ x = 23 \end{cases}$$

There were 23 coins of 50t and 42 coins of P1.

Check:

First equation LHS = 23 + 42 = 65 = RHS

Second equation LHS = 0.5 (23)+ 42 = 11.5 + 43 = 54.5 = RHS

N.B. When solving a system of equations KEEP THE SYSTEM throughout or alternatively work with numbered equations. Insist on a clear lay-out and explanations of what is done in pupils' workings.

Do not forget that pupils are to answer the question, i.e., end with a complete sentence.

Self mark exercise 2

Use the substitution method to solve the following systems of linear equations.

1. $\begin{cases} 4x + y = 14 \\ x + 5y = 13 \end{cases}$ 2. $\begin{cases} 2x - y = 5 \\ 3x - 2y = 4 \end{cases}$ 3. $\begin{cases} x - 2y = 4 \\ 5x - 6y = 18 \end{cases}$

[substitute for x]

In the following questions form the system of equations and solve using the substitution method.

4. A car travels for 3 hours at a certain average speed and then 4 hours at another average speed. The total distance covered in the seven hours is 590 km.

   If the car would have been moving for 4 hours at the first average speed and 3 hours at the second average speed, the distance covered would have been 600 km. Find the two speeds.
5. One angle in a triangle is 120°. The difference between the two other angles is 12°. What are the sizes of the angles of the triangle?
6. A P28 bill was paid using P1 and P2 coins. 18 coins were use. How many P1 and P2 coins were used?

Check your answers at the end of this unit.

Section A4: Solving simultaneous linear equations by elimination

By adding or subtracting the two equations from each other it is possible at times to form a new equation without x (or y).

We say one of the variables is eliminated. This happens when x (or y) in the two equations have the same or opposite coefficient.

Step 1: (i) In the given equations either x or y have the same coefficient. The equations can be subtracted immediately. (ii) In the given equations either x or y have opposite coefficients. The equations can be added immediately.

Example 1

Onkabetse bought 2 pens and an exercise book at the school shop and paid P 4.80. Dimakatso bought two pens and two exercise books and paid P8.00. They did not know the price of a pen and the price of an exercise book, but decided to work it out. They called the price of a pen x thebe and the price of an exercise book y thebe. The system to be solved is

$$\begin{cases} 2x + y = 480 & (1) \\ 2x + 2y = 800 & (2) \end{cases}$$

Subtracting equation (2) and equation (1) gives

(2) – (1) : $y = 320$ (elimination of x)

Substituting this value in equation (1): $2x + 320 = 480$

$$2x + 320 - 320 = 480 - 320$$

$$2x = 160$$

$$x = 80$$

An exercise book cost P3.20 and a pen P0.80.

CHECK

Substituting in (1): LHS = 2(80) + 320 = 160 + 320 = 480 = RHS (correct)

Substituting in (2): LHS = 2(80) + 2(320) = 160 + 640 = 800 = RHS (correct)

Step 2:

If the system to be solved is such that neither the coefficients of x or y in the two equations are the same or opposite then appropriate multiplication of one (or both) equations can ensure the coefficients of x (or y) becoming equal (or equal and opposite). Then addition / subtraction can be done to eliminate one of the variables.

Example 2

At a petrol pump Mrs. Tiroyane paid P 43 for 3 tins of oil and 25 litres of petrol. For 2 similar tins of oil and 15 litres of the same type of petrol Miss Sebokolodi had to pay P26.40. At what price is the petrol pump owner selling oil and petrol?

Taking the price of a tin of oil to be P x and the price of petrol P y the system of equations to be solved is

$$\begin{cases} 3x + 25y = 43 & (1) \\ 2x + 15y = 26.40 & (2) \end{cases}$$

As the coefficients of x nor y are identical, adding or subtracting will NOT eliminate any of the two variables.

You can now decide to eliminate either x or y by multiplying both equations such that the coefficients do become the same.

To eliminate x multiply equation (1) by 2 and equation (2) by 3

$$\begin{cases} 3x + 25y = 43 & (1) \times 2 \\ 2x + 15y = 26.40 & (2) \times 3 \end{cases}$$

$$\begin{cases} 6x + 50y = 86 & (1) \\ 6x + 45y = 79.20 & (2) \end{cases}$$

Subtracting equation (1) and equation (2) gives

(1) – (2): $5y = 6.80$ (elimination of x)

$$\frac{5y}{5} = \frac{6.80}{5}$$

$$y = 1.36$$

Substituting this value in equation (1):

$$3x + 25 \cdot 1.36 = 43$$

$$3x + 34 = 43$$

$$3x = 9 \quad \text{(subtracting 34 from both sides)}$$

$$x = 3$$

An tin of oil costs P 3.00 and 1 litre of petrol costs P1.36.

CHECK

Substituting in (1): LHS = 3(3) + 25(1.36) = 9 + 34 = 43 = RHS (correct)

Substituting in (2): LHS = 2(3) + 15(1.36) = 6 + 20.40 = 26.40 = RHS (correct)

Self mark exercise 3

Use the elimination method to solve the following systems of linear equations.

1. $\begin{cases} 4x + y = 14 \\ x + 5y = 13 \end{cases}$ 2. $\begin{cases} 2x - y = 5 \\ 3x - 2y = 4 \end{cases}$ 3. $\begin{cases} x - 2y = 4 \\ 5x - 6y = 18 \end{cases}$

In the following questions form the system of equations and solve using the elimination method.

4. To cover a distance of 10 km Ntsholetsang ran for 20 minutes and jogged for 40 minutes. If he would have jogged 20 minutes and run for 40 minutes he would have covered 1 km more. With what average speed is he running?
5. Five years ago a father was four times as old as his daughter is now. The sum of their ages is now 50. How old is the daughter now?

Tickets for a school concert are differently priced for adults and for pupils. In each of the following questions, find the cost of an adult's ticket and of a pupil's ticket:

6. 2 adults' tickets and 3 pupils' tickets cost P 7, while 2 adults' and 4 pupils' tickets cost P8.
7. Price for 2 adults and 6 pupils is P 24, and price for 2 adults and 4 children is P19.
8. Price for 1 adult and 2 pupils is P6, and price for 2 adults and 2 pupils is P8.

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9. Investigate the number of solutions a simultaneous system of linear equations can have.

$$\begin{cases} ax + by = c \\ Ax + By = C \end{cases}$$

Consider in your investigation these systems.

a) $\begin{cases} x + 2y = 10 \\ 3x + 6y = 18 \end{cases}$ b) $\begin{cases} 2x - -3y = 5 \\ 6x - 9y = 15 \end{cases}$

What is the relationship between a, b, c and A, B, C in each case?

Check your answers at the end of this unit.

Unit 4, Practice task

1. Discuss
   (i) advantages and disadvantages of each of the three methods for solving simultaneous linear equations.
   (ii) when a particular method is more appropriate to use than one of the others.
2. Which method would you use to answer the following questions? Justify your choice.
   a) 4 pens and one book cost P54. 2 pens and 2 books cost P72. Write down a pair of simultaneous equations and solve them to find the price of a pen and of a book.
   b) Two types of video cassette have different playing times. Two of type 1 and one of type 2 play for 13 hours. Three of type 1 and two of type 2 play for 22 h.Write down a pair of simultaneous equations and solve them to find the playing time of cassettes of type 1 and type 2.
   c) Wool is sold in balls and in skeins each having a different length of wool. Two balls and three skeins have a length of 30 m, while 4 balls and 2 skeins have a length of 28 m. Write down a pair of simultaneous equations and solve them to find what length of wool is in a ball and what length in a skein.
   d) Running for 30 min and then jogging for 20 min covers a distance of 11.5 km. Jogging 30 min and running 20 min covers a distance of 11 km. What is the average running and jogging speed?
3. Design and try out activities you would set your pupils in order to cover the objective "Pupils should be able to form and solve systems of linear simultaneous equations" ensuring that opportunity for learning is given to the low and high achievers.

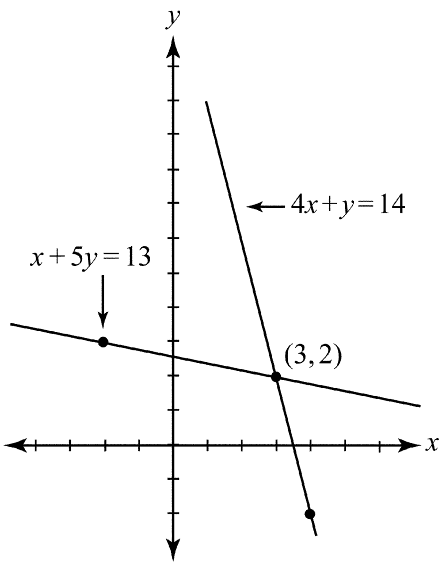
   Write an evaluative report.

Present your assignment to your supervisor or study group for discussion.

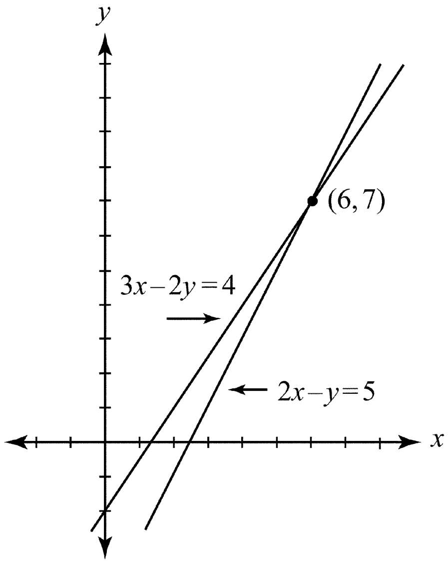
Unit 4: Answers to self mark exercises

Self mark exercise 1

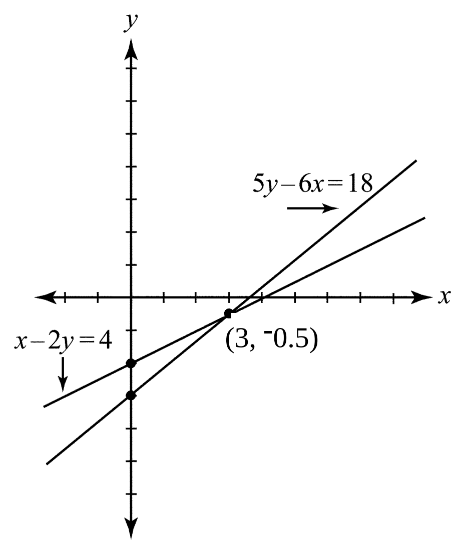
1. (3, 2)



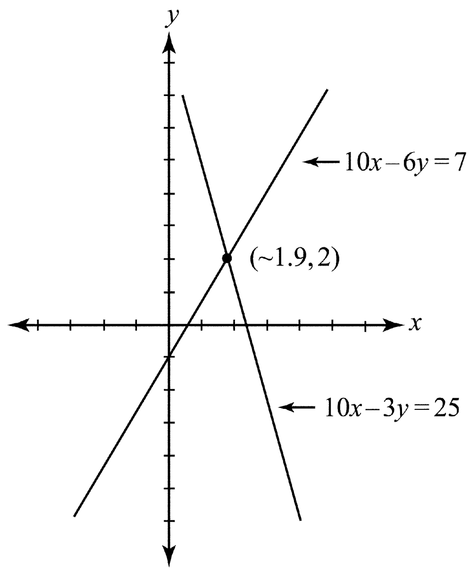
2. (6, 7)



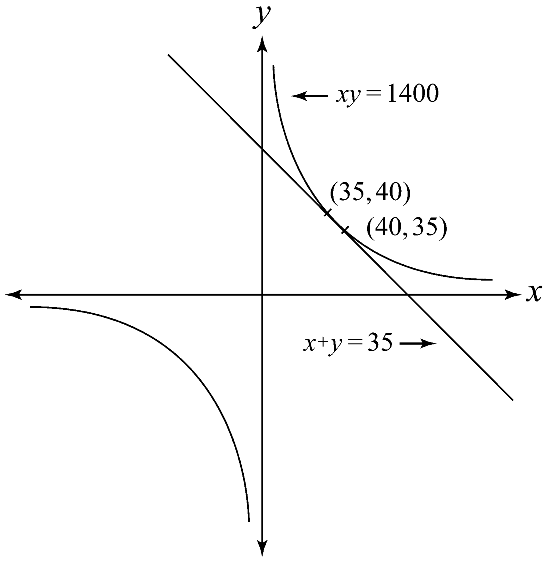
3. (3, $^{-}$0.5)



4. (1.9, 2)



5. (35, 40), (40, 35)



Note: the graph does not clearly show the two solutions.

Self mark exercise 2

1. (3, 2) 2. (6, 7) 3. (3, -0.5)

4. 80 km/h, 90 km/h 5. 24°, 36°, 120° 6. 8 × P1, 10 × P2

Self mark exercise 3

1. (3, 2) 2. (6, 7) 3. (3, -0.5) 4. 12km/h 5. 9 years

6. adults P2.-, pupils P 1.- 7. adults P4.50, pupils P 2.50

8. adults P2.-, pupils P 2.-

9. Assuming non zero coefficients

   No solution (dependent system) if $\frac{a}{A} = \frac{b}{B} \neq \frac{c}{C}$

   1 solution if $\frac{a}{A} \neq \frac{b}{B}$

   infinitely many solutions if $\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$

Unit 5: Cubic equations

Introduction to Unit 5

Solution of polynomial equations of the form $a_0 + a_1x + a_2x^2 + ... + a_nx^n = 0$, $a_n \neq 0$ have been at the centre of the historical development of the solution of equations. Reference to cubic equations is found in the time of the Babylonian civilisation (1800 - 1600 BC) when advances to solve the (general) equations were made. The final questions related to the cubic equations were finally solved by Euler in 1732. The quartic equation was solved partly by Ferrari (1540) by reducing it to a cubic equation, which can now be solved. Since Euler's (1707 - 1783) days, solving polynomial equations of higher order was doomed to failure; however, only in the nineteenth century did mathematicians succeed in proving that polynomial equations of order five and higher cannot be solved by expressing the roots in the coefficients (Ruffini, 1814; Abel, 1824, Galois, 1831).

Purpose of Unit 5

In this unit you will extend your knowledge on the solving of polynomial equations to the first steps in the solution of cubic equations. A complete solution, involving complex numbers, can be found in more advanced books on mathematics.

Higher order equations rarely appear in any workplace outside engineering, so there is no attempt to supply practical examples. History is used instead.

It is not suggested that cubic equations will form a part of your junior secondary teaching. But your brighter students may explore higher order equations anyhow; this material may equip you to encourage or challenge them.

Objectives

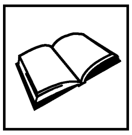
When you have completed this unit you should be able to:

- recall historical facts related to the solution of (cubic) equations
- solve simple cubic equations (finding one real solution)

Time

To study this unit will take you about 5 hours.

Unit 5: Cubic Equations



Section A: Equations in historical perspective

Algebraic solution of equations is a topic nearly all mathematicians have considered at one time or the other. The history of mathematics indicates that general solutions to polynomial equations up to the degree of four can be obtained. Polynomial equation of degree five and higher have no algebraic solution.

Reference to cubic equations is made as early as 1800 -1600 B.C. when on Babylonian clay tablets tables of cubes and cube roots are found. A table of values of $n^2 + n^3$ has also been found. Such tables can be used to solve cubic equations of special format.

For example, to solve $2x^3 + 3x^2 = 540$, the Babylonians worked as follows (using our notation):

Multiply by 4	$8x^3 + 12x^2 = 2160$
Substitute $y = 2x$	$y^3 + 3y^2 = 2160$
Substitute $y = 3z$	$27z^3 + 27z^2 = 2160$
Divide by 27	$z^3 + z^2 = 80$

From tables $z = 4$, hence $y = 12$ and $x = 6$ is a solution to the original equation.

The Ancient Greeks were very much interested in geometry including volume of solids which lead to equations of degree three. They found geometrical solutions by intersecting conics (c. 400 BC). Omar Khayyam (AD 1100) extended the Greek methods and classified thirteen special types of cubic equations that have positive roots and could be solved.

The next rapid development of the algebraic solution of cubic equations and algebra in general was around 1500 in Italy (the time of the Renaissance) and was due to

(i) the facility in handling numerical computation by using the Hindu-Arabic numeral system, which was superior to the Roman numeral system.

(ii) the invention of printing with movable type (c. 1450) which stimulated standardisation in the use of symbols to allow communication with a wider audience.

(iii) economic growth and 'world' trade which supported academic activities and stimulated the exchange of ideas with scholars in other parts of the then known (to European) world.

In 1515 Scipione del Ferro (1465 - 1526), a professor of mathematics at the university of Bologna (Italy), developed a solution for the cubic equation $x^3 + ax = b$. He did not publish the result as was customary among mathematicians of that time. He kept the method secret in order to have an advantage in mathematical duels and tournaments. These contests were very important as they influenced the appointment of university staff. Scipione del Ferro disclosed the solution method to one of his students (Antonio Maria Fior of Venice) only just before his death. Fior let it be known that he was able to solve cubic equations of the form $x^3 + ax = b$.

In 1535 Fior challenged a prominent mathematician of those days, Nicollo Tartaglia (1500 -1557, then teaching in Venice), to a contest as he did not believe Tartaglia's claim that he could solve cubic equations of the format $x^3 + ax^2 = b$. A few days before the contest he also managed to solve the cubic equations of the format $x^3 + ax = b$. Each contestant set thirty questions for the other to solve within a given time. Tartaglia solved all the questions set by Fior, while Fior failed to solve even a single one set by Tartaglia. Tartaglia obviously won the contest as he could now solve two types of cubic equations while Fior managed only one type.

Tartaglia wished to establish his reputation by publishing his method in a major work of algebra he planned to write. Girolamo Cardano (1501 - 1576, professor at Milan and Bologna), learning of Tartaglia's success, was very much interested in the method. Tartaglia finally informed Cardano of his method in 1539, swearing Cardano to secrecy until he had finished his publication. The reason that Tartaglia informed Cardano was the hope for a prospective patron, as at that time Tartaglia was without a substantial source of income. Perhaps this was due to his speech impediment (Tartaglia means stammerer) obtained as a child from a sabre cut in the fall of Brescia to the French in 1512. Cardano learned about Ferro's solution prior to Tartaglia's and might have felt this released him from the solemn oath.

In 1545 Girolamo Cardano (1501 - 1576) published his *Ars Magna* containing Scipione del Ferro's solution to the cubic equation (now generally known as Cardan's formula, Cardan being the English name for Cardano) and Ludovico Ferrari's (1522 - 1565) solution to the quartic equation. Ferrari was a pupil of Cardano and solved the quartic equation in 1540 set as a challenging problem by his teacher.

To solve $x^3 + 6x = 20$ Cardano finds two numbers p and q such that $p - q = 20$ (the constant term) and $pq = 8$ (the cube of one third of the coefficient of x). The two equations in p and q can be solved by using substitution methods, leading to a quadratic equation in p (or q). A solution to the cubic equations is then $x = \sqrt[3]{p} - \sqrt[3]{q}$.

Cardano's method applies to any cubic equation after being transformed to remove the term in x^2.

Cardano would have made more progress if imaginary numbers had not been so puzzling to him, as he left unanswered question such as: What should be done with negative and imaginary roots? What should be done (in the so called irreducible case) when Cardano's method produced apparently imaginary expressions like $x = \sqrt[3]{81 + 30\sqrt{-3}} + \sqrt[3]{81 - 30\sqrt{-3}}$ which in fact represents the number $^{-}6$ a solution to the cubic equation $x^3 - 63x = 162$?

Carl Friedrich Gauss (1777 - 1855) at the age of 20 published a theorem which he called the fundamental theorem of algebra: **Every algebraic equation of degree *n* has *n* roots.**

The method of Cardano gives one root of the cubic equations. In cases where the equation has one real and two imaginary roots the real root is obtained. In cases of three real roots Cardano's method is problematic.

Self mark exercise 1

1. The Babylonian table for $n^2 + n^3 = 150$ has solution $n = 5$.

   Use the Babylonian method to solve the cubic equation $5x^3 + 2x^2 = 48$
2. Solve using Cardano's method the cubic equations

   a) $x^3 + 6x = 20$

   b) $x^3 + 9x = 32$

Check your answers at the end of this unit.

Section B: Solving the cubic equation

(i) Graphical solution of cubic equations

In unit 3 one of the methods used to solve quadratic equations was graphical. The method is not restricted to solving quadratic equations, but is applicable to all equations and is often a first step in solving equations that cannot be solved algebraically. This method—with graphic calculators becoming more and more a common tool in the mathematics classroom—will win in importance in years to come. Without graphic tools available the method is work intensive—the drawing of a table with values of x and y and the plotting of the corresponding graph is time consuming. On the other hand the method allows practice of a wide range of important concepts and techniques, such as substitution, evaluation of expressions, plotting of graphs, rearranging equations, relating solution of equations to intersecting of graphs, relating number of (real) possible solutions of an equation ($y = f(x) = 0$) to the number of intersections of the graph of $y = f(x)$ with the x-axis.

The cubic equation $ax^3 + bx^2 + cx + d = 0$ can be solved by plotting the graph of the function $y = f(x) = ax^3 + bx^2 + cx + d$ and considering where the curve cuts the x-axis (with equation $y = 0$). That is, you find the x-coordinate of the point of intersection of $y = ax^3 + bx^2 + cx + d$ with $y = 0$.

Alternatively, you have the possibility of rearranging the equation and finding x-coordinates of points of intersections of other graphs.

For example:

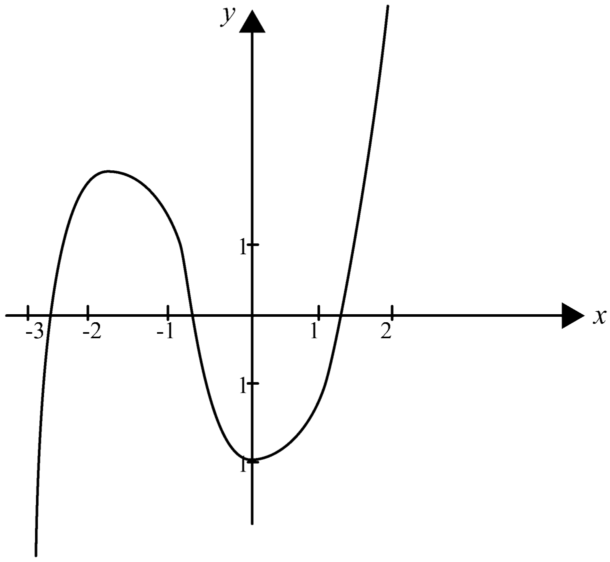
Solve $x^3 + 2x^2 - 2x - 2 = 0$ graphically.

Method 1:

Draw a table with values for x and the corresponding values for y.

x	$^-4$	$^-3$	$^-2$	$^-1$	0	1	2	3	4
y	$^-26$	$^-5$	2	1	$^-2$	$^-1$	10	37	86

Plot the graph on 2 mm graph paper.



Read from your graph the x-coordinates of the points where the curve cuts the x-axis. This will give $x = {}^-2.5$ (1 dp), $x = {}^-0.7$ (1 dp) and $x = 1.2$ (1 dp).

Method 2:

Rearranging the given equation to, for example, $x^3 + 2x^2 = 2x + 2$ and plotting the graphs of $y = f(x) = x^3 + 2x^2$ and of $y = g(x) = 2x + 2$.

The x-coordinates of the points of intersection are the solutions to the equation $x^3 + 2x^2 = 2x + 2$ and hence to the equation $x^3 + 2x^2 - 2x - 2 = 0$.

Method 3:

Rearranging the given equation as $x^3 = {}^-2x^2 + 2x + 2$ and next plotting the graphs of $f(x) = y = x^3$ and of $y = g(x) = {}^-2x^2 + 2x + 2$.

The x-coordinates of the points of intersection of the two graphs will give approximate values to the solution of the equation $x^3 + 2x^2 - 2x - 2 = 0$.

Self mark exercise 2

1. Solve $x^3 + 2x^2 - 2x - 2 = 0$ graphically by drawing the graphs of

   a) $y = f(x) = x^3 + 2x^2$ and $y = g(x) = 2x + 2$

   b) $f(x) = y = x^3$ and $y = g(x) = {}^{-}2x^2 + 2x + 2$.

   c) Compare the three graphs you have now drawn to solve the same equation. Which do you consider to give the best result in the easiest way?

2. Solve the following equations graphically using two different arrangements of the equation

   a) $2x^3 - 3x = 6$ b) $x^3 - 5x + 3 = 0$

3. Investigate the graphs of $f(x) = ax^3 + bx^2 + cx + d$ for different values of the constants. How many 'different' type of cubic graphs are there? How many roots can the equation

   $ax^3 + bx^2 + cx + d = 0$ have?

Check your answers at the end of this unit.

(ii) Solving cubic equations by trial and improvement method

Graphical methods will, in nearly all cases, lead to approximate solutions to the equation. That is why the graphical method is frequently used to obtain a first approximation to the roots of the equation, and next trial and improvement is used to obtain the root(s) to whatever accuracy is required.

For example: One of the roots of the equation $x^3 + 2x^2 - 2x - 2 = 0$ was graphically found to be approximately 1.2.

Taking this as starting value the root can be improved by trail and error to whatever number of decimal places may be required.

x	$x^3 + 2x^2 - 2x - 2$	
1.2	0.208	too large (from looking at graph)
1.1	$^{-}0.449$	too small
1.16	$^{-}0.068$	too small
1.17	$^{-}0.0006$	too small
1.175	0.0335	too large
1.172	0.0130	too large
1.171	0.0062	too large
1.1705	0.0028	too large
1.1701	0.00009	too large
1.17005	$^{-}0.0002$	too small

To 3 decimal places the root is 1.170.

Self mark exercise 3

1. Solve by trial and improvement method the equation $x^3 – 5x – 4.5 = 0$ to 3 significant figures.
2. Find correct to 2 decimal places the root between $x = 0$ and $x = 1$ of the equation $3x^3 – 2x^2 – 9x + 2 = 0$.
3. Find correct to 3 significant figures the root of $x^3 + x – 11 = 0$.

Check your answers at the end of this unit.

(iii) Solving cubic equations by factorisation

Some cubic equations (and higher order polynomial equations) can be solved by factorising the equations. In the case of the cubic equations, as long as one of the roots is integral, factorisation can help to solve the equation. To obtain the integral root the factor theorem (module 4, unit 5) is used.

For example:

(1) Solve for x the equation $x^3 + 2x^2 – x – 2 = 0$.

Try the factors of the constant ($^{-}2$): $\pm 1, \pm 2$.

If $P(x) = x^3 + 2x^2 – x – 2$, then $P(1) = 1 + 2 – 1 – 2 = 0$, hence $(x – 1)$ is a factor.

$P(x) = x^3 + 2x^2 – x – 2 = (x – 1)(x^2 + x + 2)$ by long division

$= (x – 1)(x + 2)(x + 1)(x + 2)$ factorisation of the quadratic.

Hence $x^3 + 2x^2 – x – 2 = 0$

$(x – 1)(x + 2)(x + 1)(x + 2) = 0$

$x – 1 = 0$ or $x + 1 = 0$ or $x + 2 = 0$

$x = 1$, $x = -1$, $x = {}^{-}2$ are the roots of the cubic equation.

(2) Solve for x the equation $x^3 + 4x^2 + 4x + 3 = 0$.

Try the factors of 3: $\pm 1, \pm 3$ (N.B. trying the positive values is not very useful as the expression has only positive coefficients!)

If $P(x) = x^3 + 4x^2 + 4x + 3$, then $P(^{-}1) = {}^{-}1 + 4 – 4 + 3 \neq 0$,
$(x + 1)$ is NOT a factor.

$P(^{-}3) = {}^{-}27 + 36 – 12 + 3 = 0$, hence $(x + 3)$ is a factor.

$x^3 + 4x^2 + 4x + 3 = (x + 3)(x^2 + x + 1)$ by long division

The quadratic does not factorise further
(discriminant $D = b^2 – 4ac = 1 – 4 = {}^{-}3 < 0$).

$x^3 + 4x^2 + 4x + 3 = 0$

$(x + 3)(x^2 + x + 1) = 0$

$x + 3 = 0$ or $x^2 + x + 1 = 0$ (no real solutions)

$x = {}^-3$.

The cubic equation has a single real solution $x = {}^-3$.

Self mark exercise 4

1. a) Use the factor theorem to find a factor of $3x^3 - x^2 - 38x - 24$.

   b) Using factorisation solve $3x^3 - x^2 - 38x - 24 = 0$.

2. Given that the following cubic equations have (at least) one integer solution, solve the equations by using the factor theorem and factorisation.

   a) $x^3 + 3x^2 - 10x - 24 = 0$ b) $x^3 + 6x^2 - 4x - 24 = 0$

   c) $x^3 + 2x^2 - x - 2 = 0$ d) $6x^3 - 17x^2 - 5x - 6 = 0$

Check your answers at the end of this unit.

(iv) Solve cubic equations algebraically

The method described below is Cardano's or Cardan's method. The formula you will end up with for solving cubic equations is the Cardano's formulae for solving cubic equations. ***Note: The algebra in what follows becomes quite involved. If you find it lacks usefulness for your teaching situation, skip ahead to Unit 5 Assignment 1.***

Step I Changing the given cubic equation to the reduced or normal format.

To solve the cubic equation $Ax^3 + Bx^2 + Cx + D = 0$ the first step is to reduce the equation to a cubic equation without a quadratic term. You can always make the coefficient of x^3 equal to 1 by dividing throughout by A. This gives the equation $x^3 + \frac{B}{A}x^2 + \frac{C}{A}x + \frac{D}{A} = 0$. We can hence restrict our discussion to equations of the format $x^3 + ax^2 + bx + c = 0$. This equation can be reduced to an equation without a term in x^2.

The substitution $x = z - \frac{a}{3}$ will accomplish this. The equation reduces to $z^3 + pz + q = 0$; this is called the reduced or normal cubic equation, where $p = b - \frac{1}{3}a^2$ and $q = \frac{2}{27}a^3 - \frac{1}{3}ab + c$.

Do the substitution and the simplification to check whether the stated values for p and q are correct. To work out $(z - \frac{a}{3})^3$ you use the binomial theorem (Pascal's triangle for the coefficients).

If you have any problem you can go through the working below.

$-\frac{a}{3}$ Substituting $x = z - \frac{a}{3}$ in $x^3 + ax^2 + bx + c = 0$ will give:

$$(z-\frac{a}{3})^3 + a(z-\frac{a}{3})^2 + b(z-\frac{a}{3}) + c = 0$$

$$z^3 + 3z^2(-\frac{a}{3}) + 3z(-\frac{a}{3})^2 + (-\frac{a}{3})^3 + a(z^2 - \frac{2a}{3}z + \frac{a^2}{9}) - bz - \frac{1}{3}ab + c = 0$$

$$z^3 - az^2 + \frac{1}{3}a^2z - \frac{1}{27}a^3 + az^2 - \frac{2}{3}a^2z + \frac{1}{9}a^3 - bz - \frac{1}{3}ab + c = 0$$

$$z^3 + z(\frac{1}{3}a^2 - \frac{2}{3}a^2 + b) + (-\frac{1}{27}a^3 + \frac{1}{9}a^3 - \frac{1}{3}ab + c) = 0$$

$$z^3 + z(b - \frac{1}{3}a^2) + (\frac{2}{27}a^3 - \frac{1}{3}ab + c) = 0$$

Hence taking $p = b - \frac{1}{3}a^2$ and $q = \frac{2}{27}a^3 - \frac{1}{3}ab + c$ the equation is reduced to $z^3 + pz + q = 0$

Example: $x^3 - 3x^2 + x + 6 = 0$. Here a = $^-3$, $b = 1$, $c = 6$

Substitute $x = z + 1$

$$(z+1)^3 - 3(z+1)^2 + (z+1) + 6 = 0$$

$$z^3 + 3z^2 + 3z + 1 - 3[z^2 + 2z + 1] + z + 1 + 6 = 0$$

$$z^3 - 2z + 5 = 0$$

Check that the stated values for p and q in terms of a, b and c indeed give $^-2$ and 5.

Step II Solve the reduced cubic equation.

To solve the cubic equation $z^3 + pz + q = 0$ the following identity is considered:

$$(u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3 \quad \text{(binomial expansion)}$$

$$= u^3 + v^3 + 3uv(u+v) \quad \text{(reordering)}$$

$$(u+v)^3 - 3uv(u+v) - u^3 - v^3 = 0 \quad \text{(reordering)}$$

Comparing this with the reduced cubic equation $z^3 + pz + q = 0$ we obtain

$z = u + v$ $\quad$ $^-3uv = p$ and $q = {}^-u^3 - v^3$

From these three equations you can see that to find z (and from z we can find x from the original equation) you are to find the values u and v such that

$^-3uv = p$ and $q = {}^-u^3 - v^3$

Substituting $v = \frac{-p}{3u}$ from the first equation in the second gives:

$$q = {}^-u^3 - \left(\frac{-p}{3u}\right)^3$$

$$u^3 - \frac{p^3}{27u^3} + q = 0$$

$$27u^6 - p^3 + 27u^3 q = 0$$

$$27u^6 + 27u^3 q - p^3 = 0$$

This is a quadratic equation in u^3 and can be solved by using the quadratic formula.

Let's take $u^3 = t$

Then the quadratic equation to be solved is $27t^2 + 27qt - p^3 = 0$

$$u^3 = t = \frac{-27q \ \pm \sqrt{(27q)^2 + 4 \times 27 \times p^3}}{2 \times 27}$$

$$= \frac{-27q \ \pm 27\sqrt{q^2 + \frac{4}{27}p^3}}{2 \times 27}$$

$$= \frac{-q \ \pm \ \sqrt{q^2 + \frac{4}{27}p^3}}{2}$$

$$= -\frac{1}{2}q \ \pm \frac{1}{2}\sqrt{q^2 + \frac{4}{27}p^3}$$

$$= -\frac{1}{2}q \ \pm \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}$$

As $v^3 = {}^-u^3 - q$

$$\mathit{If}\ u^3 = -\frac{1}{2}q + \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}$$

$$v^3 = -\left(-\frac{1}{2}q + \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}\right) - q$$

$$= \ -\frac{1}{2}q - \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}$$

Taking for u^3 the value with the negative sign for the square root would give for v^3 the similar expression but with a positive sign before the square root.

Due to the symmetry in u and v of the two equations
${}^-3uv = p$ and $q = {}^-u^3 - v^3$, the roots of the quadratic equation can be randomly given to be either u^3 or v^3.

As you have now the values of u^3 and v^3, the values of u and v are the cube root from these expressions.

As $z = u + v$ one solution to the cubic equation is obtained (Cardan's formula for the cubic).

$$z = u + v = \sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}} + \sqrt[3]{-\frac{1}{2}q - \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}}$$

Work through the following examples carefully. The working below does not show all the steps so you have to fill in 'the missing' ones yourself.

(1) Solving $x^3 + 3x^2 + 9x - 13 = 0$

$a = 3$, $b = 9$, $c = {}^-13$

Substitute $x = z - \dfrac{a}{3} = z - 1$ leads to the reduced equation

$z^3 + 6z - 20 = 0$ (check this).

Hence $p = 6$, $q = {}^-20$.

One solution to the equation as given by Cardan's formula is therefore substituting p = 6 and

$q = {}^-20$

$$\begin{aligned} z &= \sqrt[3]{10 + \sqrt{100 + 8}} + \sqrt[3]{10 - \sqrt{100 + 8}} \\ &= \sqrt[3]{10 + \sqrt{108}} + \sqrt[3]{10 - \sqrt{108}} \qquad (108 = 3 \times 36) \\ &= \sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}} \\ &= \sqrt[3]{1 + 3\sqrt{3} + 9 + 3\sqrt{3}} + \sqrt[3]{1 - 3\sqrt{3} + 9 - 3\sqrt{3}} \\ &= \sqrt[3]{(1 + \sqrt{3})^3} + \sqrt[3]{(1 - \sqrt{3})^3} \qquad (\textit{check that } (1 + \sqrt{3})^3 = 1 + 3\sqrt{3} + 9 + 3\sqrt{3}) \\ &= 1 + \sqrt{3} + 1 - \sqrt{3} \\ &= 2 \end{aligned}$$

$$\begin{aligned} x &= z - 1 \\ &= 2 - 1 = 1 \end{aligned}$$

Hence one root of $x^3 + 3x^2 + 9x - 13 = 0$ is $x = 1$.
This implies $(x - 1)$ is a factor.

Factorising $x^3 + 3x^2 + 9x - 13 = (x - 1)(x^2 + 4x + 13)$ by long division

The quadratic has no other real roots as the discriminant $D = 16 - 4 \times 13 < 0$

(2) Solving $x^3 - 12x + 16 = 0$.

This equation is already in the reduced or normal form.

$p = {}^-12$ and $q = 16$ substituted into Cardan's formula gives

$$\begin{aligned} x &= \sqrt[3]{{}^-8 + \sqrt{64 - 64}} + \sqrt[3]{{}^-8 - \sqrt{64 - 64}} \\ &= \sqrt[3]{{}^-8} + \sqrt[3]{-8} = {}^-2 - 2 = {}^-4 \end{aligned}$$

Hence $x + 4$ is factor of $x^3 - 12x + 16$

$x^3 - 12x + 16 = (x + 4)(x^2 - 4x + 4)$ by long division

$= (x + 4)(x - 2)(x - 2)$ factorisation of the quadratic

Solving $x^3 - 12x + 16 = 0$

$(x + 4)(x - 2)(x - 2) = 0$

$x + 4 = 0$ or $x - 2 = 0$

$x = {}^{-}4$ or $x = 2$.

Looking at Cardan's formula

$$z = u + v = \sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}} + \sqrt[3]{-\frac{1}{2}q - \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}}$$

one can already deduce when 'problems' will arise. If we work with real numbers only the square root from a negative number has no meaning—or is undefined over the real numbers. The above two examples made the expression under the square root $\frac{1}{4}q^2 + \frac{1}{27}p^3$ either positive (example 1) or equal to zero (example 2). In these cases an exact value for the cube root (as demonstrated in example 1) can be found. However when the expression $\frac{1}{4}q^2 + \frac{1}{27}p^3$ is negative a problem arises. This case is called the irreducible case as shown in the next example

(3) Solving $x^3 - 6x - 4 = 0$ using Cardan's formula with $p = {}^{-}6$ and $q = {}^{-}4$ gives:

$$x = \sqrt[3]{2 + \sqrt{4 - 8}} + \sqrt[3]{2 - \sqrt{4 - 8}}$$

$$= \sqrt[3]{2 + \sqrt{-4}} + \sqrt[3]{2 - \sqrt{-4}} = \sqrt[3]{2 + 2i} + \sqrt[3]{2 - 2i} \quad \text{(taking } \sqrt{-1} = i\text{)}$$

Square roots from negative numbers are called complex numbers. Cardan's formula leads to cube roots of complex numbers and there is no general algebraic or arithmetic method for finding the exact value of these cube roots. That is the reason to call this case: irreducible. Cardan's formula is of little use in this case. The solution to this case needs use of complex numbers and trigonometry.

Self mark exercise 5

1. Solve the cubic equations using Cardano's method.

   a) $x^3 - 5x^2 + 9x - 9 = 0$

   b) $x^3 + 6x^2 + 3x - 10 = 0$

   c) $x^3 - 6x^2 + 10x - 8 = 0$

Check your answers at the end of this unit.

Unit 5, Assignment 1

1. The Cardan formula is of great logical significance, but it is not very useful for applications. The method of successive approximations (trial and improvement method) is of greater importance (in combination with graphical methods to find a 'starting' value).

   Discuss the above statement. Justify that algebraic solutions of cubic and quartic equations are rarely included in algebra courses while methods of trial and improvement are.

2. Could any part of this unit be used in the classroom or mathematics club? Justify your answer.

Present your assignment to your supervisor or study group for discussion

Module 5, Practice activity

1. a) List the knowledge and skills you expect pupils in junior secondary school to acquire through solving of equations.

   b) Describe how you create opportunities in your class for pupils to acquire the knowledge and skills you listed in 1a.

2. a) To consolidate forming and solving of linear equations play "Guess my number" game. For example "I think of a number and multiply the number by 2. I subtract 6 from the answer and divide the result I obtained by 2. My answer is 9. What was the number I was thinking of?"

   The teacher might state the questions and pupils (in groups) come up with an answer and write their answer down (large) on paper. After some time groups can show their answers by holding up their papers. Points are awarded to each group.

   Alternatively groups can make the questions to be answered by the other groups. A point system can be designed.

   Try out the activity and write a report.

   b) Play "I will tell the answer you got."

   The teacher present questions such as "All of you think of a 2 digit number. Multiply your number by 2. Add 10 to the answer. Multiply your answer again by 10. Divide the number you got by 4. Subtract the number you started with. I will tell you your answer: 5."

   Ask pupils to explain how you could know.

   Try out the activity and write an evaluative report.

3. a) Design a game to consolidate the solving of quadratic equations.

   b) Try it out and write an evaluative report.

Continued on next page

Continued from previous page

4. a) Design a guided worksheet for pupils to 'discover' the quadratic formula.

   b) Try out the worksheet(s) with your pupils.

   c) Write a critical evaluation of the activity.

5. "When solving equations skill of manipulations (use of standard algorithms) are important. Drill and practice are only invigorating and stimulating if they have meaning and purpose; otherwise they can be stultifying and boring."

   Discuss the statement and indicate the implications for teaching solving of equations.

6. This module deliberately pays attention to historical development of mathematics. This is to make you aware that mathematics is a human creation and ever changing. It is also important that pupils in the classroom become aware of this.

   Design a classroom lesson in which you use the history of mathematics. Try it out and write an evaluative report. Include a paragraph on 'The use of the history of mathematics in classroom teaching at junior secondary level'.

7. Did this module lead to any changes in the methods you use to assist pupils in the learning of solving of equations?

   If yes, list the changes and explain why you decided to make a change.

   If no, explain why .

Present your assignment to your supervisor or study group for discussion.

Summary

You have come to the end of this module. I hope you found it useful, found some enjoyment in going through the module, but above all that it has helped you to improve and renovate your classroom practice.

You should have strengthened your knowledge on pupils' difficulties in learning to solve equations and how these difficulties might be avoided by using realistic examples and investigative methods.

You should have gained confidence in creating a learning environment for your pupils in which they can

(i) acquire with understanding, knowledge of solving linear and quadratic equations using a variety of methods.

(ii) consolidate through games the solving of simple linear equations.

(iii) acquire knowledge on solving systems of simultaneous linear equations.

(iv) model situations leading to linear or quadratic equations.

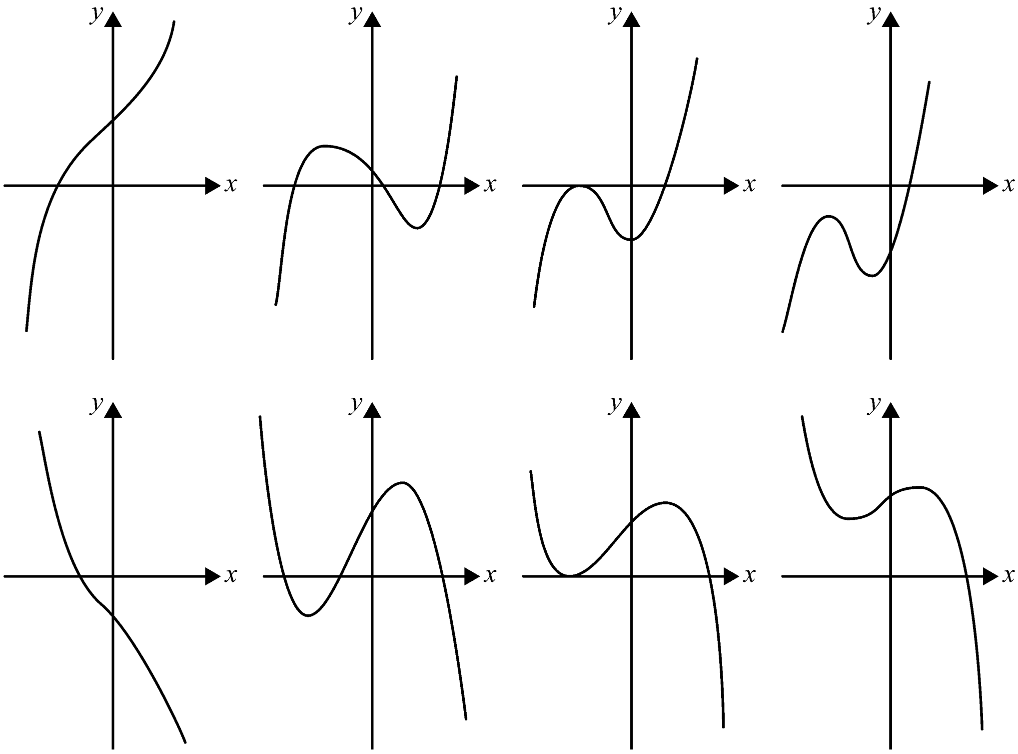
You may also have extended your content knowledge by working out the solution of cubic equations.

Unit 5: Answers to self mark exercises

Self mark exercise 1

1. $x = 2$
2. a) $x = \sqrt[3]{\sqrt{108}+10} - \sqrt[3]{\sqrt{108}-10}$ b) $\sqrt[3]{\sqrt{283}+16} - \sqrt[3]{\sqrt{283}-16}$

3 The graphs of the cubic function fall into 4 basic types.



Type 1: $a > 0$, graph of the function is increasing, has always one point of intersection with the x-axis. There are no minima / maxima.

Type 2: $a > 0$, graph of the function is increasing for large negative / large positive values of x, and has a maximum and a minimum. Graph might have 3, 2 (touching x-axis at one point) or 1 point of intersection with the x-axis.

Type 3: $a < 0$, graph of the function is decreasing, and has always one point of intersection with the x-axis. There are no maxima or minima.

Type 4: $a < 0$, graph of the function is decreasing for large negative / large positive values of x, and has a minimum and a maximum. Graph might intersect the x-axis in 3, 2 (touching axis in one point) or 1 point(s).

A cubic equation has always at least one real root. It might have three or two different roots.

Self mark exercise 2

1. ab) $^{-}2.5$, $^{-}0.7$, 1.2
2. a) 1.78

   b) $^{-}2.49$, 0.66, 1.83

Self mark exercise 3

1. 2.56
2. 0.22
3. 2.07

Self mark exercise 4

1. a) $(x + 3)$ and $(x - 4)$ are factors. It factorises as $(x + 3)(x - 4)(3x + 2)$.

   b) $x = {}^{-}3$, $x = -\frac{2}{3}$, $x = 4$
2. a) $^{-}4$, $^{-}2$, 3

   b) $^{-}6$, $^{-}2$, 2

   c) $^{-}1$

   d) $^{-}3$, $-\frac{1}{2}$, $\frac{2}{3}$

Self mark exercise 5

1. 3
2. $^{-}5$, $^{-}2$, 1
3. 4

References

Botswana, Republic of. *Three-Year Junior Secondary Syllabus Mathematics*, 1996.

Hart, K. *Children's Understanding of Mathematics 11 - 16*, 1981,
ISBN 071 953 772X

Additional References

In preparing the materials included in this module we have borrowed ideas extensively from other sources and in some cases used activities almost intact as examples of good practice. As we have been using several of the ideas, included in this module, in teacher training over the past 5 years the original source of the ideas cannot be traced in some cases. The main sources are listed below.

NCTM *Algebra in a Technological World*, 1995, ISBN 087 353 3267

NCTM *Learning and Teaching Algebra*, 1984, ISBN 087 353 2686

NCTM *Historical Topics for the Mathematics Classroom*, 1989,
ISBN 087 353 2813

Open University, *Preparing to Teach Equations*, PM 753C,
ISBN 033 517 4396

UB-INSET *Algebra for Every One, 1996*, University of Botswana

Further reading

The Maths in Action book series are for use in the classroom using a constructivist, activity based approach, including problem solving, investigations, games and challenges in line with the ideas in this module.

OUP/Educational Book Service, Gaborone, *Maths in Action Book 1*
ISBN 019 571776 7

OUP/Educational Book Service, Gaborone, *Maths in Action Book 2*
ISBN 019 .....

OUP/Educational Book Service, Gaborone, *Maths in Action Teacher's File Book 1* ISBN 019 ...

OUP/Educational Book Service, Gaborone, *Maths in Action Teacher's File Book 2* ISBN 019

G Glossary

Algebra	the branch of mathematics that deals with the general properties of numbers and generalisations of relationships among numbers
Discriminant	the expression $b^2 - 4ac$ obtained from the quadratic equation $ax^2 + bx + c = 0$
Equation	a statement that two mathematical expression are equal. The expressions involve at least one variable
Equivalence	two mathematical statements or expressions A and B that are either both true or both false. Notation used: $A \equiv B$ for expressions, $A \Leftrightarrow B$ for equivalent statements
Identity	A statement that two mathematical expressions are equal for all values of their variables
Linear equation	equation of the form $ax + b = 0$, $a \neq 0$ and a and b are real numbers [when considering one variable only]
Open statement	statement that might be true or false depending on the situations / variables
Parameter	a constant or variable that distinguishes special cases of a general mathematical expression or formula. For example in the intercept-gradient equation of a line $y = axc + b$, a and b are parameters representing the gradient and y-intercept of any specific line
Quadratic equation	equation of the form $a_0 + a_1x + a_2x^2 = 0$, $a_2 \neq 0$ [when considering one variable only]
Polynomial equation	equation of the form $a_0 + a_1x + a_2x^2 + ... + a_nx^n = 0$, $a_n \neq 0$ with the coefficient being real numbers [when considering one variable only]
Reduced cubic equation	a cubic equation of the form $z^3 + pz + q = 0$, i.e., without a term in z^2
Root of an equation	a number which when substituted for the variable in a given equation satisfies the equation
Solution of equations	the process of finding the root(s) of a given equation
Square root	the square root of any positive number N is the positive number which when squared gives N

Statement	an expression (in words, numerals or algebraic) which might be true or false
Symmetric relation	a binary relation R is symmetric on a set if for all elements x and y of the set, xRy implies yRx, e.g., 'cousin' on the set of people and 'parallel to' on the set of lines are symmetric relations
Transitive relation	a binary relation R is transitive on a set if for all elements x, y, and z of the set, it holds that xRy and yRz implies xRz, e.g., 'greater than' 'is equal to' are transitive relations
Variable	representation of a changing quantity

Appendix 1: 2 mm squared graph paper